Information about Limits from Stewart Section 1.4

MATH 2301 (Barsamian)

Limit Laws

Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$ when n is any positive integer.
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ when n is any positive integer. (With the additional requirement that if n is even, then a must be positive. That is, $a > 0$.)
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ when n is any positive integer. (With the additional requirement that if n is even, then $\lim_{x \to a} f(x)$must be positive. That is, $\lim_{x \to a} f(x) > 0$.)

Direct Substitution Property

If f is a polynomial, or if f is a rational function with the number a in its domain, then

$$\lim_{x \to a} f(x) = f(a)$$

Missing Theorem About Limits of Ratios

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, with $\lim_{x \to a} f(x) \neq 0$ and $\lim_{x \to a} g(x) = 0$,

then the ordinary limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ *does not exist* (using the **ordinary definition of limits**).

(This situation gets revisited in Section 1.6 when we study **limits involving infinity**.)

Missing Terminology About Certain Kinds of Limits of Ratios:

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then the $\lim_{x\to a} \frac{f(x)}{g(x)}$ is said to be in $\frac{\mathbf{0}}{\mathbf{0}}$ ***indeterminate form***.

It is not possible to say whether the limit exists, and what its value might be, without first somehow converting the limit to an equivalent limit expression that is *not* in indeterminate form.

Useful Fact

If $f(x) = g(x)$ when $x = a$, then $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ match.

That is, either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, or both limits do not exist.

Theorem [2] (The One-Sided Limit Test)

The limit $\lim_{x\to a} f(x) = L$ if and only if f passes this three-part test.

1. The left limit $\lim_{x\to a^-} f(x)$ exists.
2. The right limit $\lim_{x\to a^+} f(x)$ exists.
3. The limits in (1) and (2) match, with common value L. That is, $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Theorem [3] About Limits of Two Functions Where One Function Is Bounded by The Other

If $f(x) \leq g(x)$ when x is near a (except possibly at a), and if the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

Theorem [4] The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at32158 a) and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then

$$\lim_{x\to a} g(x) = L$$

Equation [6]

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$