MATH 2301 (Barsamian) Meeting #3 (Wed Aug 30)

- Pick up handout on front table
- Sit in bottom half of room
- Sign In

<u>This week</u>

- Work on Homework
    - write it down on <u>paper</u>
    - type answers into WebAssign
- I strongly recommend buying a loose-leaf print copy of the Textbook at <u>College Bookstore</u>, uptown.

Meeting Part 1 Continue Discussion of Section 1.3 Limits

[Example 1] (Similar to Exercises 1.3 #7, 10)

Sketch graph of a function f that has all these properties

(x, y) locations implicated

$$\lim_{x \to 3^+} f(x) = 4$$ (3, 4)

$$\lim_{x \to 3^-} f(x) = 2$$ (3, 2)

$$\lim_{x \to -2} f(x) = 2$$ (-2, 2)

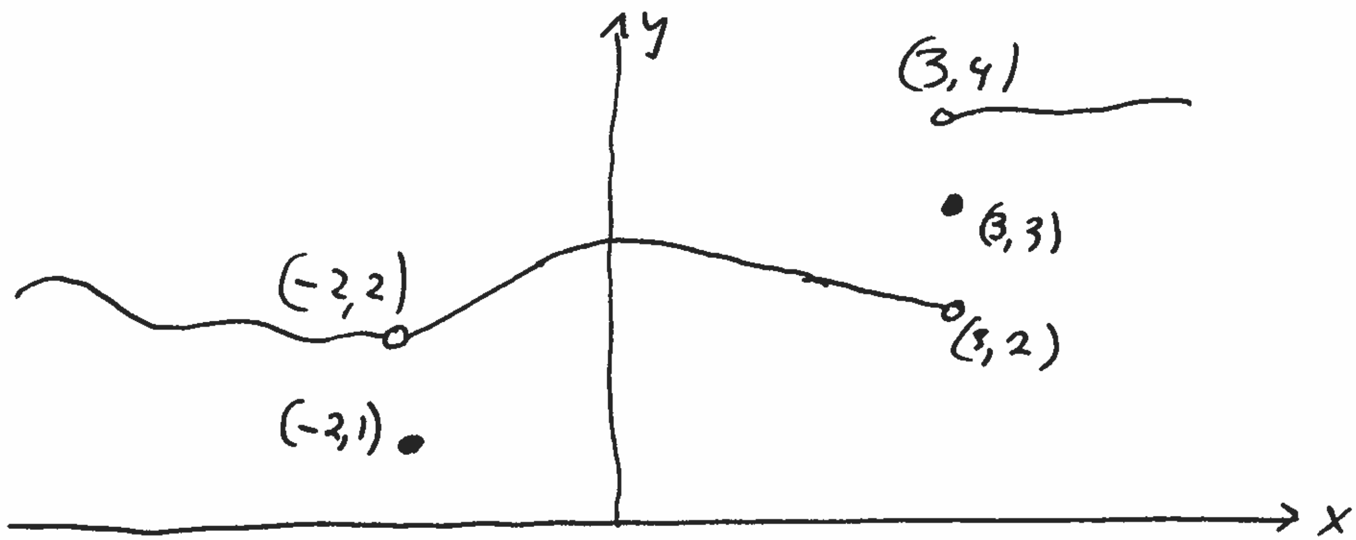
$$f(3) = 3$$ (3, 3)

$$f(-2) = 1$$ (-2, 1)

Solution

Identify the (x, y) locations that are implicated in the given properties.

Plot them on one set of axes, with open circles



Then add features to satisfy the required conditions.

[Example #2] (related to 1.3 #11 and later 1.4 #12)

let $f(x) = \dfrac{x^2 - 4x}{x^2 - 3x - 4} = \dfrac{x(x-4)}{(x+1)(x-4)}$

standard form — factored form

(a) Find $f(2)$ and $f(4)$

Solution

$$f(2) = \frac{2\cancel{(2-4)}}{(2+1)\cancel{(2-4)}} = \frac{2}{2+1} = \frac{2}{3}$$

use factored form

can cancel $\frac{2-4}{2-4}$ because $2-4 \neq 0$

$$f(4) = \frac{4(4-4)}{(4+1)(4-4)} = \frac{4 \cdot 0}{5 \cdot 0} = \frac{0}{0} = \text{undefined}$$

(b) Estimate $\lim_{x \to 2} f(x)$ and $\lim_{x \to 4} f(x)$ by making tables of (x, y) values

Solution

Start with $\lim_{x \to 4} f(x)$

x	$f(x) = \frac{x(x-4)}{(x+1)(x-4)}$
3.9	$f(3.9) = \frac{3.9\cancel{(3.9-4)}}{(3.9+1)\cancel{(3.9-4)}} = \frac{3.9}{4.9} \approx 0.795918$
3.99	$f(3.99) = \frac{3.99\cancel{(3.99-4)}}{(3.99+1)\cancel{(3.99-4)}} = \frac{3.99}{4.99} \approx 0.799599$
3.999	$f(3.999) = \dots = \frac{3.999}{4.999} \approx 0.799960$

trend $x \to 4^-$ trend $y \to 4/5$ estimate $\lim_{x \to 4^-} f(x) = \frac{4}{5}$

x	$f(x) = \frac{x(x-4)}{(x+1)(x-4)}$
4.1	$f(4.1) \approx 0.8039216$
4.01	$f(4.01) \approx 0.8003992$
4.001	$f(4.001) \approx 0.800040$

trend $x \to 4^+$ trend $y \to 4/5$ estimate $\lim_{x \to 4^+} f(x) \approx \frac{4}{5}$

Based on these tables, I would estimate $\lim_{x \to 4} \frac{x(x-4)}{(x+1)(x-4)} \approx \frac{4}{5}$

Similarly, with tables, we would <u>estimate</u>

$$\lim_{x \to 2^-} f(x) \approx 2/3$$

$$\lim_{x \to 2^+} f(x) \approx 2/3$$

$$\lim_{x \to 2} f(x) \approx 2/3$$

<u>Recap</u>

from part (a) $f(2) = 2/3$ and $f(4)$ DNE

from part (b) $\lim_{x \to 2} f(x) \approx 2/3$ and $\lim_{x \to 4} f(x) \approx \frac{4}{5}$

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[Example #3] (Similar to 1.3 #15 and related to 1.4 #21)

Consider the expression $\dfrac{\sqrt{25+h}-5}{h}$

ⓐ find value of expression when $h=0$

Solution $\dfrac{\sqrt{25+(0)}-5}{(0)} = \dfrac{\sqrt{25}-5}{0} = \dfrac{5-5}{0} = \dfrac{0}{0}$ undefined

ⓑ Use tables of values to estimate the limit $\lim\limits_{h\to 0} \dfrac{\sqrt{25+h}-5}{h}$

Solution

h	$y = \dfrac{\sqrt{25+h}-5}{h}$
-0.1	$y \approx 0.1001002$
-0.01	$y \approx 0.10001$
-0.001	$y \approx 0.100001$

trend $h \to 0^-$ trend $y \to 1/10$

h	
0.1	$y \approx 0.0999002$
0.01	$y \approx 0.0999900002$
0.001	$y \approx 0.099999$

trend $h \to 0^+$ trend: $y \to \frac{1}{10}$

So estimate $\lim\limits_{h\to 0} \dfrac{\sqrt{25+h}-5}{h} \approx \dfrac{1}{10}$

[Example #4]

estimate $\lim_{x \to 0} \frac{1}{x^2}$ by making table of values

See book example 8 page 31

trend: x getting closer + closer to 0 (but not equal to 0)

trend: y getting more + more positive, without bound.

Since y is not getting closer + closer to a number,

we say $\lim_{x \to 0} \frac{1}{x^2}$ Does not Exist according to definition of limit in Section 1.3

Later, in section 1.6, we will revisit the definition of limit. We will enlarge the definition of limit. This limit will get a new result then.

Meeting Part 2 Section 1.4 Calculating Limits

Estimating limits with (x, y) tables is tedious, requires computer in many cases, and only results in an estimate

Natural Question Is there a way to find limits exactly, by doing some sort of analysis?

Good News: Yes, using the "Precise Definition of Limit" on p.31
Very cool, beautiful techniques!

Bad News: Above the level of MATH 2301. Really hard techniques.

Good News: Results of those hard techniques can be articulated as theorems that we can use in MATH 2301

See Information about Limits handout.

Information about Limits from Stewart Section 1.4

MATH 2301 (Barsamian)

Limit Laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ when n is any positive integer.
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ when n is any positive integer. (With the additional requirement that if n is even, then a must be positive. That is, $a > 0$.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ when n is any positive integer. (With the additional requirement that if n is even, then $\lim_{x\to a} f(x)$must be positive. That is, $\lim_{x\to a} f(x) > 0$.)

Direct Substitution Property

If f is a polynomial, or if f is a rational function with the number a in its domain, then

$$\lim_{x\to a} f(x) = f(a)$$

Missing Theorem About Limits of Ratios

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, with $\lim_{x\to a} f(x) \neq 0$ and $\lim_{x\to a} g(x) = 0$,

then the ordinary limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ *does not exist* (using the **ordinary definition of limits**).

(This situation gets revisited in Section 1.6 when we study **limits involving infinity**.)

Missing Terminology About Certain Kinds of Limits of Ratios:

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then the $\lim_{x\to a} \frac{f(x)}{g(x)}$ is said to be in $\frac{0}{0}$ ***indeterminate form***.

It is not possible to say whether the limit exists, and what its value might be, without first somehow converting the limit to an equivalent limit expression that is *not* in indeterminate form.

Useful Fact

If $f(x) = g(x)$ when $x = a$, then $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ match.

That is, either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, or both limits do not exist.

Theorem [2] (The One-Sided Limit Test)

The limit $\lim_{x\to a} f(x) = L$ if and only if f passes this three-part test.

1. The left limit $\lim_{x\to a^-} f(x)$ exists.
2. The right limit $\lim_{x\to a^+} f(x)$ exists.
3. The limits in (1) and (2) match, with common value L. That is, $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Theorem [3] About Limits of Two Functions Where One Function Is Bounded by The Other

If $f(x) \le g(x)$ when x is near a (except possibly at a), and if the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist,

then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Theorem [4] The Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at32158 a) and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then

$$\lim_{x\to a} g(x) = L$$

Equation [6]

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

[Example 5] (revisiting [Example 2])

for $f(x) = \frac{x^2-4x}{x^2-3x-4}$, find $\lim_{x\to 2} f(x)$ using the limit laws.

Solution

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 2} \frac{x(x-4)}{(x+1)(x-4)} = \frac{2\cancel{(2-4)}}{(2+1)\cancel{(2-4)}} = \frac{2}{3}$$

replace f(x) with the expression for f(x)

rational function with x=2 in domain

Direct Sub property can sub x=2

this matches the result that we got in [Example 2]

— End of Meeting —