MATH 2301 (Barsamian) Lecture #6 (Meeting #8) Mon Sep 11, 2023

Sign In

Things to remember this week

Recitation Assignments ~~are~~ have been posted for tomorrow,
Tues Sep 12. (See Calendar on Course WebPage)

- Prepare before you come to recitation
- Recitation Solutions are graded.

Quiz Q2 this coming Friday Sep 15

If you like print books, get one at College Bookstore

Binders will be replenished either today or tomorrow

Friday, we were discussing $f(x) = \dfrac{x^2-3x+2}{x-4} = \dfrac{(x-1)(x-2)}{(x-4)}$

using a table of values, we estimated the following limits (using the Section 1.6 Expanded Definition of Limit, incorporating concept of infinity

we found

$$\lim_{x\to 4^+} f(x) = \infty$$ using table of values

$$\lim_{x\to 4^-} f(x) = -\infty$$ using table of values

$$\lim_{x\to 4} f(x)$$ DNE because left & right limits don't match.

Contrast these results with the results that we would have gotten using the Original Definition of limit, from Sections 1.3 and 1.4.

for $f(x) = \frac{x^2-3x+2}{x-4} = \frac{(x-1)(x-2)}{(x-4)}$

observe that $\lim_{x\to 4}$ numerator $= 6$ (for left, right, and two-sided limits)

and $\lim_{x\to 4}$ denominator $= 0$ (also for left, right, and two-sided limits)

Therefore the "Missing theorem about Limits of Ratios" on the Handout Information about Limits from Section 1.4 tells us that

$$\left.\begin{array}{l} \lim_{x\to 4^+} f(x) \text{ DNE} \\ \lim_{x\to 4^-} f(x) \text{ DNE} \\ \lim_{x\to 4} f(x) \text{ DNE} \end{array}\right\} \text{all by the result of that missing theorem}$$

In other words, we have <u>expanded</u> the <u>Definition of Limit</u> in Section 1.6, to include the concept of <u>infinity</u>

Analytical Technique for computing $\lim_{x\to 4^+} f(x)$

involving comparing the relative sizes of numerator + denominator.

this term will be slightly greater than 3

this term will be slightly greater than 2

So the numerator will be close to 6

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{(x-1)(x-2)}{(x-4)}$$

this symbol tells us that x is slightly greater than 4

So the denominator is slightly greater than 0

$$\frac{\text{number close to 6}}{\text{positive number close to 0}}$$

ratio will be huge and positive

So the limit will be infinity

$$= \infty$$

Limit from the left

$$\lim_{x\to 4^-} f(x) = \lim_{\underbrace{x\to 4^-}_{x\text{ slightly less than }4}} \frac{\overbrace{\overbrace{(x-1)}^{\text{close to }3}\overbrace{(x-2)}^{\text{close to }2}}^{\text{numerator close to }6}}{\underbrace{(x-4)}_{\text{slightly less than }0}}$$

ratio will be huge and negative number

$$= -\infty$$

$\lim_{x\to 4} f(x)$ DNE because left & right limits don't match

Consider Anthony's, Clepatra's, and Bob's solutions

Anthony: $\lim_{x\to 4^+} f(x) = \frac{(4-1)(4-2)}{4-4} = \frac{3\cdot 2}{0} = \frac{6}{0} = \infty$

$\lim_{x\to 4^-} f(x) = \frac{(4-1)(4-2)}{(4-4)} = \frac{3\cdot 2}{0} = \frac{6}{0} = \infty$

$\lim_{x\to 4} f(x) = \frac{(4-1)(4-2)}{(4-4)} = \frac{6}{0} = \infty$

Observe: Invalid Method and incorrect results!

Clepatra: $\lim_{x\to 4^+} f(x) = \frac{6}{0}$ DNE

$\lim_{x\to 4^-} f(x) = \frac{6}{0}$ DNE

$\lim_{x\to 4} f(x) = \frac{6}{0}$ DNE

Observe: Invalid Method and Incorrect results

Bob $\lim_{x\to 4^+} f(x) = \frac{(4-1)(4-2)}{4-4} = \frac{3\cdot 2}{0} = \frac{6}{0} = \infty$

$\lim_{x\to 4^-} f(x) = \frac{(4-1)(4-2)}{(4-4)} = \frac{3\cdot 2}{0} = \frac{6}{0} = -\infty$

$\lim_{x\to 4} f(x) = \frac{(4-1)(4-2)}{(4-4)} = \frac{3\cdot 2}{0} = \frac{6}{0}$ DNE

Observe: Bob has the right answers, but they do not follow from his work. Indeed, he did the exact same calculation three times but gave three different results.

All three solutions of all three students are invalid solution

[Example] find $\lim_{x \to 0} \frac{1}{x} + \frac{1}{|x|}$

<u>Solution</u> We must first figure out the function $f(x) = \frac{1}{x} + \frac{1}{|x|}$

Remember that $|x|$ is a piecewise-defined function.

That is,

When $x \geq 0$, the symbol $|x|$ means just x.

But when $x < 0$, the symbol $|x|$ means $-x$.

This means that $f(x)$ is a piecewise-defined function, too.

When $x \geq 0$ $f(x) = \frac{1}{x} + \frac{1}{|x|}$ means $f(x) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$

But when $x < 0$, $f(x) = \frac{1}{x} + \frac{1}{|x|}$ means $f(x) = \frac{1}{x} + \frac{1}{-x} = \frac{1}{x} - \frac{1}{x} = 0$

So to compute $\lim_{x \to 0} f(x)$, we will have to do one-sided limits.

Limit from the right

replaced f(x) with the appropriate formula

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{2}{x}$$

the ratio will be a huge, positive number

this symbol tells us that the denominator is a positive number close to 0.

$$= +\infty$$

Limit from the Left

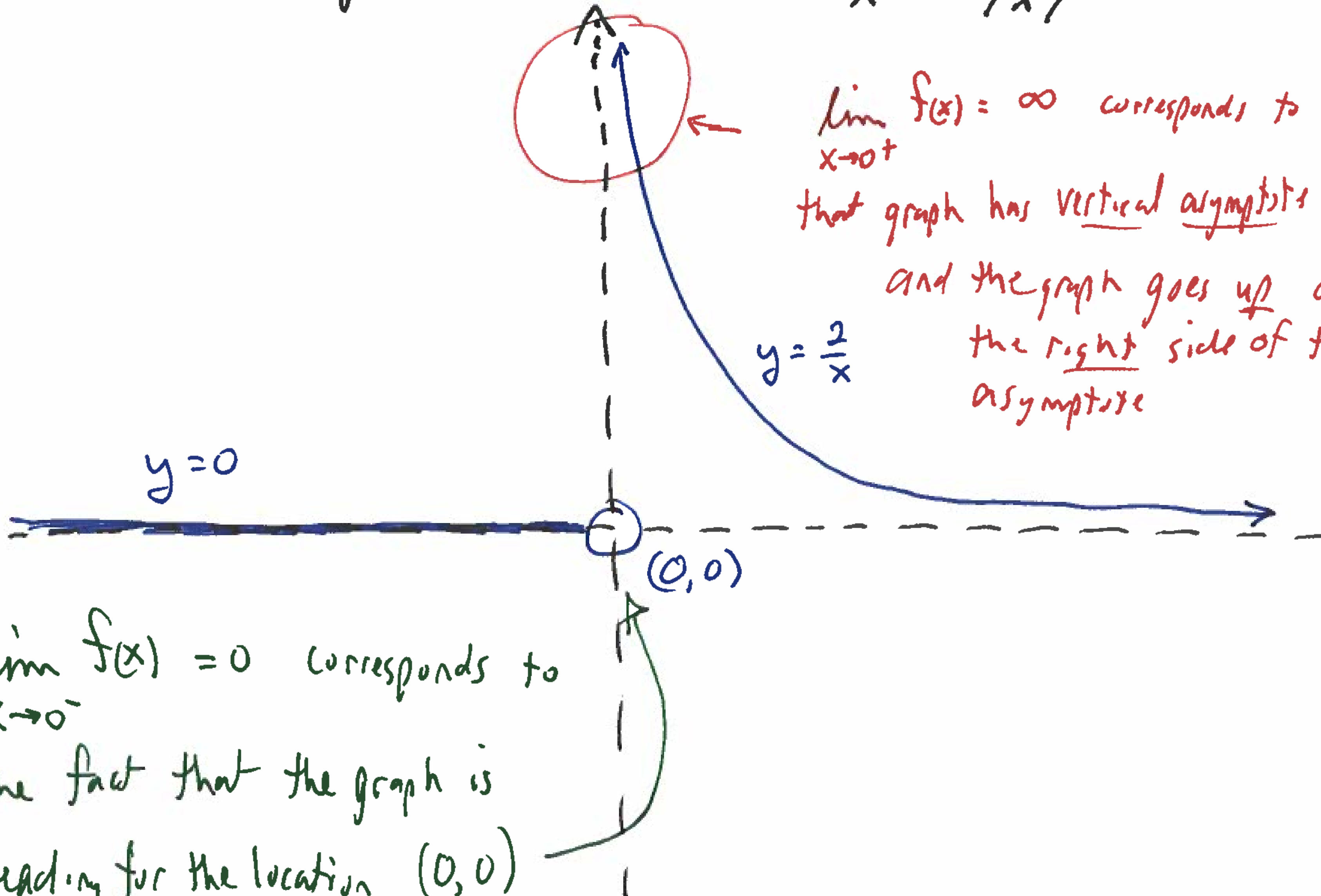
replace f(x) with the appropriate formula

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$$ by Limit Law 7

So the limit is

$\lim_{x \to 0} f(x)$ DNE because left & right limits dont match.

Consider the graph of $f(x) = \frac{1}{x} + \frac{1}{|x|}$



$\lim_{x \to 0^+} f(x) = \infty$ corresponds to the fact that graph has vertical asymptote at $y=0$, and the graph goes up along the right side of that asymptote

$\lim_{x \to 0^-} f(x) = 0$ corresponds to the fact that the graph is heading for the location $(0,0)$ from the left.

New Topic

Limits at Infinity

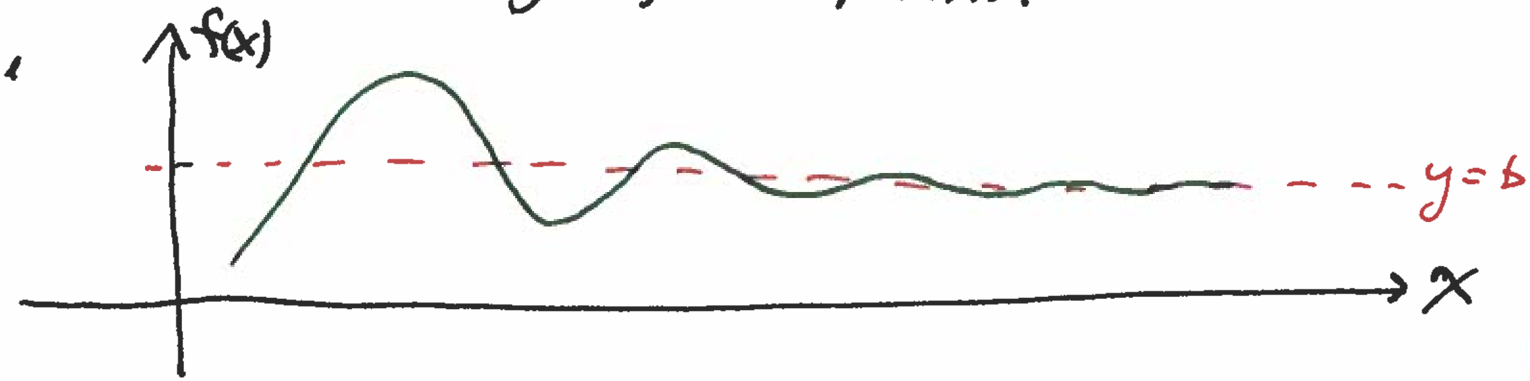
Definition of Limit at Infinity

Symbol: $\lim\limits_{x \to \infty} f(x) = b$

usage: f is function, b is a real number

meaning: The values of $f(x)$ can be made to stay arbitrarily close to $y=b$ (as close as desired) by requiring that x be sufficiently large and positive

picture



terminology: The line $y=b$ is a horizontal asymptote on the right for the graph of $f(x)$.

[Example 2] (a) Find $\lim_{x\to\infty} \dfrac{7x^3-3x^2+x}{5x^3-x+2}$

Solution $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\frac{7x^3}{x^3}-\frac{3x^2}{x^3}+\frac{x}{x^3}}{\frac{5x^3}{x^3}-\frac{x}{x^3}+\frac{2}{x^3}}$

Trick: divide all terms by the highest power of x that appears in the denominator

$= \lim_{x\to\infty} \dfrac{7-\frac{3}{x}+\frac{1}{x^2}}{5-\frac{1}{x^2}+\frac{2}{x^3}}$

$= \dfrac{\lim_{x\to\infty} 7 - \lim_{x\to\infty}\frac{3}{x} + \lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 5 - \lim_{x\to\infty}\frac{1}{x^2} + \lim_{x\to\infty}\frac{2}{x^3}}$

Because all the separate limits exist, limit law #5 and limit law #1 tell us that we can just find each limit separately

(see explanations of limits on next page)

$= \dfrac{7-0+0}{5-0+0}$

$= \dfrac{7}{5}$

(b) Explain what the result of (a) tells us about the graph of f(x)

Solution: graph of f(x) has horizontal asymptote on right with line equation $y=\frac{7}{5}$.

Explanations of limits on previous page

$\lim_{x\to\infty} 7 = 7$ and $\lim_{x\to\infty} 5 = 5$

These are by a limit law just like limit law #7, but generalized to apply to limits at infinity.

$\lim_{x\to\infty} \frac{3}{x}$ } So the ratio will be a positive number very close to 0 $= 0$

denominator is a huge positive number

So the limit will be zero

$\lim_{x\to\infty} \frac{1}{x^2}$ } So the ratio will be a positive number very close to 0 $= 0$

denominator is a huge positive number

So the limit will be zero

$\lim_{x\to\infty} \frac{2}{x^3} = 0$ by reasoning analogous to above

⑭

[Example 3] let $g(x) = \dfrac{7x^5 - 3x^2 + x}{5x^3 - x + 2}$ ← changed to 5

Find this limit

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\dfrac{7x^5}{x^3} - \dfrac{3x^2}{x^3} + \dfrac{x}{x^3}}{\dfrac{5x^3}{x^3} - \dfrac{x}{x^3} + \dfrac{2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{7x^2 - \frac{3}{x} + \frac{1}{x^2}}{5 - \frac{1}{x^2} + \frac{2}{x^3}}$$

} numerator getting more & more positive

} denominator closer and closer to 5

ratio getting huge, positive

$$= \infty$$

Right end of graph is going up!

[end of lecture]