Lecture 10 Wed Sep 20, 2023 Section 2-2. The Derivative as a Function.

Reminders:
Pick up graded work from the table in front.

Sign up.

Exam XI will be this Friday during class.

To day Finish discussion of Section 2-2.
The Derivative as a Function.

Meeting Part 1

Example 1: For a function

$f(x) = 3x^2 - 5x + 7.$

(a) Find the Derivative using the definition of the derivative.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Gets the parts:

$f(x) = 3x^2 - 5x + 7$

$f(\ ) = 3(\ )^2 - 5(\ ) + 7.$ empty version.

$$f(x+h) = 3(x+h)^2 - 5(x+h) + 7.$$
$$= 3(x+h)(x+h) - 5x - 5h + 7$$
$$= 3(x^2 + 2xh + h^2) - 5x - 5h + 7$$
$$= 3x^2 + 6xh + 3h^2 - 5x - 5h + 7$$

Build the limit and find its value.

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 7 - (3x^2 - 5x + 7)}{h}.$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{5x} - 5h + \cancel{7} - \cancel{3x^2} + \cancel{5x} - \cancel{7}}{h}.$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h}.$$

$$= \lim_{h \to 0} \frac{h(6x+3h-5)}{h}.$$

since $h \to 0$, $h \neq 0$, we can cancel $\frac{h}{h}$.

$$= \lim_{h \to 0} 6x + 3h - 5$$

substitute $h = 0$

$$= 6x + 3(0) - 5$$

$$= 6x - 5$$

$$\Rightarrow f'(x) = 6x - 5.$$

<u>Note</u>: You are not allowed to use Derivative Rules on this exam!.

(b) Find the slope of the line tangent to the graph at $x=2$.

<u>Solution</u>

The slope of the line tangent to the graph at $x = 2$ is

$m = f'(2)$.

$m = f'(2) = 6(2) - 5 = 12 - 5 = 7$.

(c) Find the equation of the line tangent to the graph at $x = 2$.

<u>solution</u>:

We need to build this equation.

$$(y - f(a)) = f'(a)(x - a).$$

$a = 2$. (x coordinate of point of tangency)

$$f(a) = f(2) = 3(2)^2 - 5(2) + 7$$

$$= 3(4) - 10 + 7$$

$$= 12 - 10 + 7$$

$$= 9.$$

$f(a) = f(2) = 9$ (y coordinate of point of tangency).

$f'(a) = f'(2) = 6(2) - 5 = 12 - 5 = 7.$

$\Rightarrow f'(2) = 7.$

$(y - 9) = 7(x - 2)$

$y - 9 = 7x - 14$

$y = 7x - 14 + 9$

$y = 7x - 5.$

(d) Find the slope of the line tangent to the graph at $x=0$

Solution

Set $x=0$ and find $m=f'(0)$.

$$m=f'(0)=6(0)-5=-5$$

$$m=f'(0)=-5.$$

(e) Find the x coordinates of all points on the graph that have horizontal lines.

Solution:

set $m=0$ or $f'(x)=0$. and solve for x.

$f'(x) = 6x-5 = 0$

$6x-5=0$

$6x = 5$

$x = \frac{5}{6}$.

Meeting Part 2 : How can a function fail to be differentiable?

One way: the graph of f doesn't have a point.

Example:

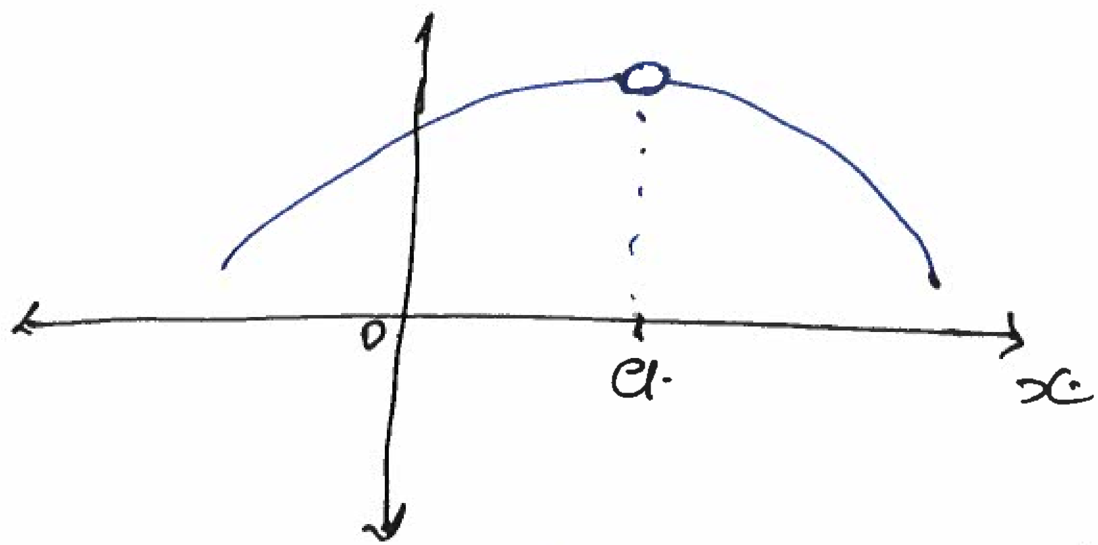


Fig 1

This graph has a discontinuity.

From Fig 1.

Observe that there cannot be a tangent line at $x = a$, because there is no point there that can serve as the point of tangency.

In general, if a function $f(x)$ does not ~~have~~ have a point at $x = a$, then $f'(a)$ will fail to exist.

Another way: The Tangent line can fail to exist.

Example: Consider $f(x) = |x|$.

Solution.

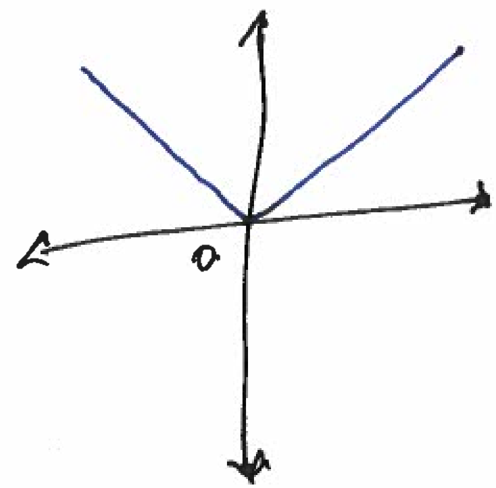
(a). If $x > 0$, then $|x| = x$.

Choose h to be small such that $x + h > 0$.

Therefore $|x+h| = x+h$.

For $x > 0$.

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}.$$



$y = f(x) = |x|$.

Fig

$$= \lim_{h \to 0} \frac{(x+h) - x}{h}.$$

$$= \lim_{h \to 0} \frac{x+h-x}{h}.$$

$$= \lim_{h \to 0} \frac{h}{h}.$$

since $h \to 0$, $h \neq 0$, so we can cancel $\frac{h}{h}$.

$$= \lim_{h \to 0} 1.$$

$$= 1.$$

$$f'(x) = 1.$$

Therefore f is differentiable for any $x > 0$.

(b) For $x < 0$, $|x| = -x$.

$x+h < 0$, Therefore $|x+h| = -(x+h)$.

For $x<0$,

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h}.$$

$$= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h}.$$

$$= \lim_{h\to 0} \frac{-x-h+x}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h}.$$

since $h\to 0$, $h \neq 0$, so we can cancel $\frac{h}{h}$.

$$= \lim_{h \to 0} -1 = -1.$$

$\Rightarrow f'(x) = -1.$

Therefore f is differentiable for any $x < 0$.

(c) For $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$ if the limit exist.

Compute the left and right limit.

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

since $h \to 0^+$, $h \neq 0$, so we can cancel $\frac{h}{h}$.

and

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0^-} -1$$

$$= -1.$$

since $h \to 0^-$, $h \neq 0$, so we can cancel $\frac{h}{h}$.

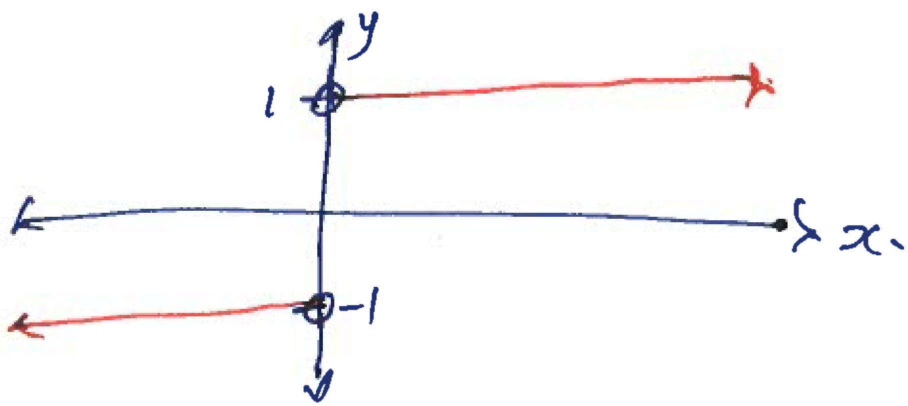
Since the limits do not match, $f'(0)$ does not exist.

Therefore $f(x) = |x|$ is differentiable at all x except $x=0$.

This is because its graph changes direction abruptly when $x=0$.

See Fig 2

$$f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$

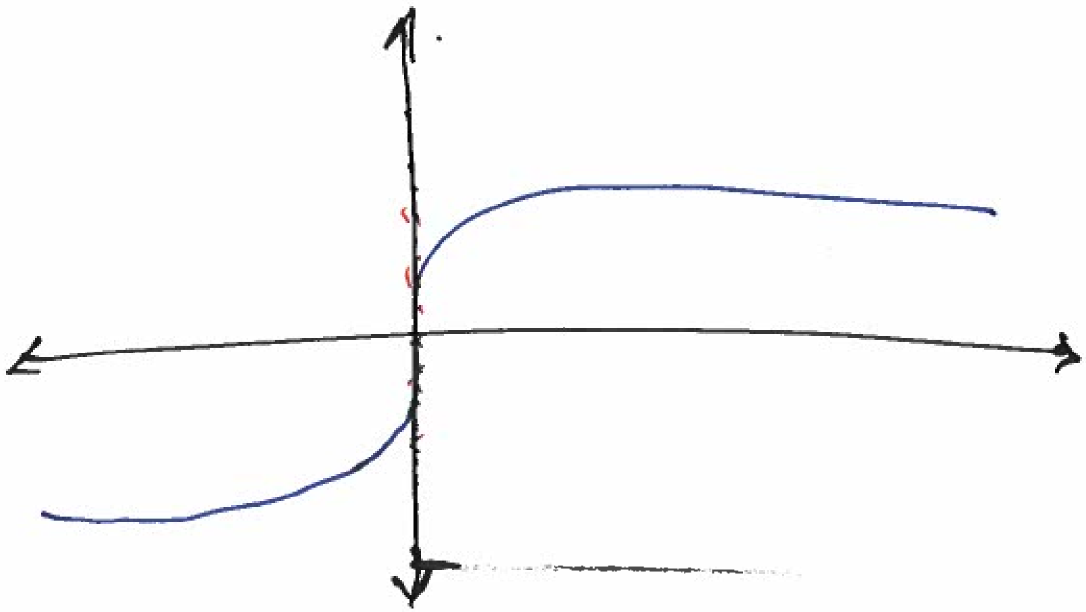


$y = f'(x)$

Note: In general, if a function $f(x)$ has a sharp cusp point at $x = a$, then $f'(x)$ will ~~fail~~ fail to exist.

Another way a function can fail to be differentiable: Tangent line exists but is vertical.

<u>Example</u>: $f(x) = y = x^{1/3}.$



We have a tangent line at $(0,0)$ and that is a vertical line.

Observe that $f'(0)$ stands for the number m that is the slope of the line that is tangent to the graph of f at $x=0$.

But since the tangent line is vertical, its slope is undefined.

Therefore, $f'(0)$ does not exist.

Note: In general, if a function $f(x)$ has a vertical tangent line at $x = a$, then $f'(a)$ will fail to exist.

Meeting Part 3: Notation for the Derivative:

We have other notation for derivatives:

$$\frac{d}{dx}$$

Example: $f(x) = 3x^2 - 5x + 7$

From part 1, Example 1, (a).

$$f'(x) = 6x - 5$$

This could also be written as

$$\frac{d}{dx} f(x) = 6x - 5.$$

OR

$$\frac{df}{dx} = 6x - 5$$

OR

$$\frac{d}{dx}(3x^2 - 5x + 7) = 6x - 5.$$

This Ends the Lecture.

$\boxtimes$