Lecture 10 Wed Sep 20, 2023

Reminders:

- Pick up quizzes from the table in front
- Sign in
- Exam X1 will be this Friday during class

Today: Finish discussion of Section 2.2.

[Example 1] For the function

$$f(x) = 3x^2 - 5x + 7$$

(a) (Exercise 2.2 # 20) Find the derivative using the <u>definition of the derivative</u>.

<u>Solution:</u> We have to find this limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Get parts:

- $f(x) = 3x^2 - 5x + 7$
- $f(\ ) = 3(\ )^2 - 5(\ ) + 7$ ← "empty version"
- $f(x+h) = 3(x+h)^2 - 5(x+h) + 7$

$$= 3(x^2 + 2xh + h^2) - 5(x+h) + 7$$

$$= 3x^2 + 6xh + 3h^2 - 5x - 5h + 7$$

Build limit and evaluate:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 7 - (3x^2 - 5x + 7)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 7 - 3x^2 + 5x - 7}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h}$$

$$= \lim_{h \to 0} \frac{h(6x + 3h - 5)}{h}$$

Since $h \to 0$, $h \neq 0$, so we can cancel $\frac{h}{h}$

$$= \lim_{h \to 0} (6x + 3h - 5)$$

no longer indeterminate
substitute $h=0$

$$= 6x + 3(0) - 5$$

$$= 6x - 5$$

Hence, for $f(x) = 3x^2 - 5x + 7$,
we found $f'(x) = 6x - 5$

(b) Find the slope of the line tangent to the graph of $f(x)$ at $x=2$.

Solution:

This means we must find $f'(2)$

$$f'(2) = 6(2) - 5 = 12 - 5 = \boxed{7}$$

(c) Find the equation of the line tangent to the graph of $f(x)$ at $x = 2$.

Solution: This means we must build the equation

$$y - f(a) = f'(a)(x-a)$$

Get parts:

- $a = 2$ (x-coord of the point of tangency)
- $f(a) = f(2) = 3(2)^2 - 5(2) + 7$

  $= 12 - 10 + 7$

  $= 9$ (y-coord of point of tangency)
- $f'(a) = f'(2) = 7$ ← from part (b)

Build equation:

$$y - f(a) = f'(a)(x-a)$$

$$y - 9 = 7(x-2)$$

$$y - 9 = 7x - 14$$

$$\boxed{y = 7x - 5}$$

(d) Find the slope of the line tangent to the graph of $f(x)$ at $x = 0$.

Solution: $f'(0) = 6(0) - 5 = \boxed{-5}$

(e) Find the x coordinates of all points on the graph of $f(x)$ that have <u>horizontal</u> tangent lines.

<u>Solution:</u> Horizontal tangent lines have slope $m = 0$, We must set $f'(x) = 0$, and solve for x.

$f'(x) = 0$

$6x - 5 = 0$

$6x = 5$

$\boxed{x = \frac{5}{6}}$

More than one horiz. tangent line?

Ex

$f(x) = \sin(x)$

Ex

$f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

$= 3(x^2 - 1)$

$= 3(x+1)(x-1) = 0$

$x+1=0$ or $x-1=0$

$x=-1$ or $x=1$

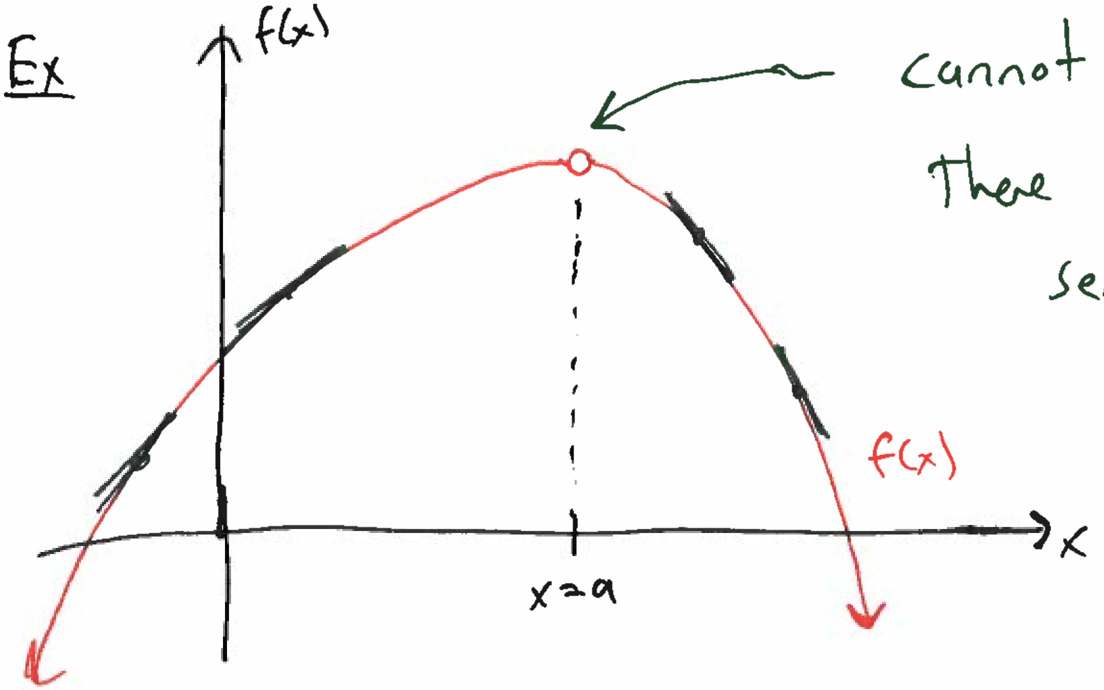
<u>Def</u>. A function $f(x)$ is <u>differentiable at $x=a$</u> if $f'(a)$ exists.

- A function $f(x)$ is <u>differentiable</u> if $f(x)$ is differentiable for all x.

<u>Question</u>: How can a function fail to be differentiable at $x=a$?

1 The graph does not have a point at $x=a$.

Ex



cannot have a tangent line here

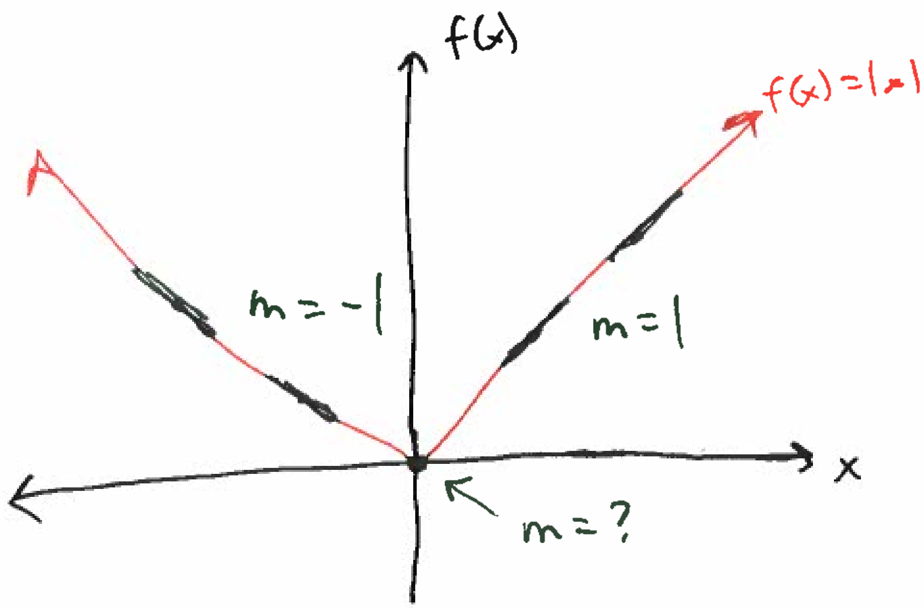
There is no point that can serve as the point of tangency.

Thus, $f'(a)$ DNE.

In general, if $f(a)$ DNE, then $f'(a)$ DNE.

② The tangent line can fail to exist.

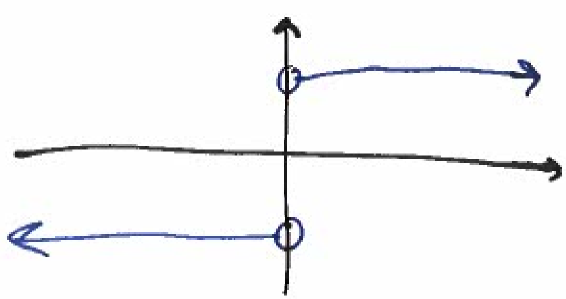
Ex $f(x) = |x|$



Observe:

- For $x > 0$, the tangent lines have slopes $m=1$
- For $x < 0$, the tangent lines have slopes $m=-1$
- For $x = 0$, there is no well-defined tangent line.

Graph of $f'(x)$

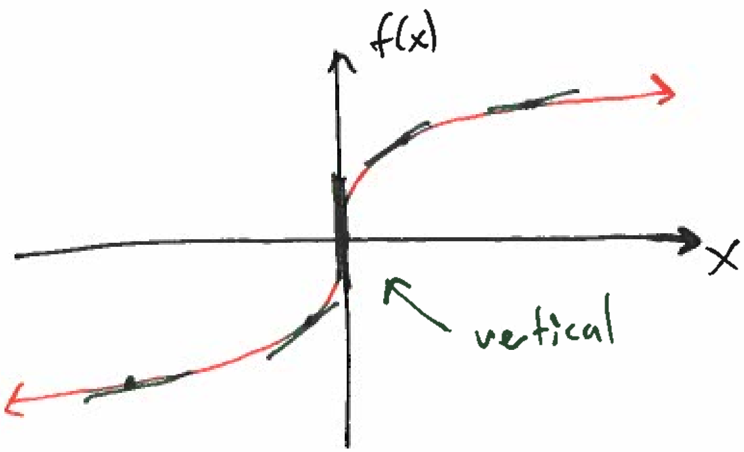


$f'(0)$ DNE

In general, if $f(x)$ has a sharp cusp at $x=a$ then $f'(a)$ DNE

(3) The tangent line is vertical

Ex $f(x) = x^{1/3}$



Remember, $f'(0)$ means the slope of the line tangent to the graph of $f(x)$ at $x=0$.

But since this tangent line is vertical, its slope is undefined.

Thus $f'(0)$ DNE.

In general, if $f(x)$ has a vertical tangent line at $x=a$, then $f'(a)$ DNE.

More notation for the derivative

Lagrange Notation

$f'(x)$, y'

Leibniz Notation

$\frac{d}{dx} f(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$

Ex $f(x) = 3x^2 - 5x + 7$

$$f'(x) = 6x - 5$$

Can also write as: $\frac{d}{dx} f(x) = 6x - 5$

$$\frac{df}{dx} = 6x - 5$$

$$\frac{d}{dx}(3x^2 - 5x + 7) = 6x - 5$$