MATH 2301 (Barsamian) Lecture #15 (Wed Oct 4, 2023)

Pick Up Graded Quizzes

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Quiz Q4 this coming Fri Oct 6 will have 4 problems, not 3
See Calendar entry for Fri Oct 6 for details

Quiz Q5 will be Wed Oct 11

Two important Comments on Quiz Q3

- Everything you write should be part of a sentence (whether in prose or in mathematical symbols). So math expression should be presented in equations with valid left sides.
  eg: $\cancel{8\cos(x)}$ $f'(x) = 8\cos(x)$
- Tangent line equation form $(y - f(a)) = f'(a)(x-a)$ is much more helpful than $y - y_1 = m(x - x_1)$

Section 2.6 Implicit Differentiation

Equations -vs- functions

$5x - y = 7$ equation involving x and y. Describes y "implicitly"

$y = 5x - 7$ equation involving x and y, solved for y in terms of x. Gives y as a function of x. Describes y "explicitly".

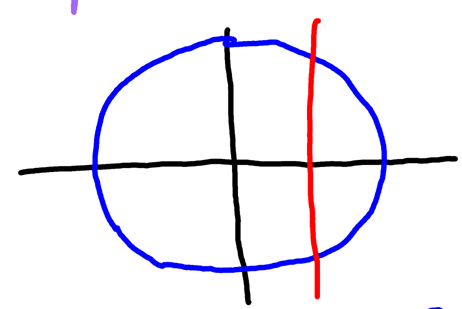
these two equations express the same relationship between x and y.

$x^3 + y^3 = 7$ equation involving x and y. Describes y "implicitly"

$y = (7 - x^3)^{1/3}$ equation involving x and y, solved for y in terms of x. Gives y as a function of x. Describes y "explicitly"

These two equations express the same relationship between x and y.

$x^2 + y^2 = 7$ equation involving x and y
expresses a relationship between x and y
describes y "implicitly".
Cannot be solved for y as a function of x.



We know this because the graph fails the vertical line test. Is not the graph of a function.

Suppose we have equation involving x & y and we want to find $\frac{dy}{dx}$

If equation can be solved for y in terms of x, we should do that first

$$y = \text{some expression involving } x$$

then take the ordinary derivative

$$\frac{dy}{dx} = \frac{d}{dx}\,\text{some expression involving } x$$

If the equation cannot be solved for y in terms of x we can still find $\frac{dy}{dx}$ using a method called Implicit Differentiation

The Method of Implicit Differentiation

Used for finding $\frac{dy}{dx}$ when x, y are related by an equation that is <u>not</u> solved for y.

Start with: equation involving x and y

<u>Step 1</u>: Find $\frac{d}{dx}$ of both sides of that equation.

Keep in mind the difference between taking derivative of x and taking the derivative of y

$$\frac{d}{dx}x = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

power function with $n=1$

power rule with $n=1$

$$\frac{d}{dx}y = y'$$ thats as far as we can go

Result of Step 1 will be a new equation involving x, y, y'

<u>Step 2</u> Solve that equation for y'

Result will be $y' =$ some expression involving x, y

[Example 1] Suppose $x^3 + y^3 = 7$

Find y' using <u>ordinary differentiation</u>

<u>Solution</u>

Solve for y in terms of x

$$x^3 + y^3 = 7$$
$$y^3 = 7 - x^3$$
$$y = (7 - x^3)^{1/3}$$

Find ordinary derivative

$$y' = \frac{d}{dx}(7 - x^3)^{1/3}$$

$$= \frac{1}{\cancel{3}(7 - x^3)^{2/3}} \cdot -\cancel{3}x^2$$

$$= -\frac{x^2}{(7 - x^3)^{2/3}}$$

<u>Chain Rule Details</u>

$\text{inner}(x) = 7 - x^3$

$\text{inner}'(x) = -3x^2$

$\text{outer}(\ ) = (\ )^{1/3}$ $\quad n = \frac{1}{3}$ ✓

$\text{outer}'(\ ) = \frac{1}{3}(\ )^{1/3 - 1}$

$= \frac{1}{3}(\ )^{-2/3} = \frac{1}{3(\ )^{2/3}}$

[Example 2] Suppose $x^3 + y^3 = 7$

Find $\frac{dy}{dx}$ using Implicit Differentiation

Solution

Start with $x^3 + y^3 = 7$

equation involving x, y

Step 1

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(7)$$

power function with $n = 3$

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 0$$

Power rule

nested function $(y)^3$

use chain rule

$$3x^2 + 3(y)^2 \cdot y' = 0$$

$$3x^2 + 3y^2 y' = 0$$

new equation involving x, y, y'

Chain Rule Details

inner(x) $= y$

inner'(x) $= y'$

outer( ) $= (\ )^3$

outer'( ) $= 3(\ )^2$

Step 2 Solve for y'

$$3y^2 y' = -3x^2$$

$$y' = -\frac{3x^2}{3y^2}$$

$$y' = -\frac{x^2}{y^2}$$

equation of the form

y' = expression involving x, y

This result doesn't look like result of [Example 1]

Rewrite derivative result of [Example 2]

from equation $x^3+y^3=7$

we can solve for $y = (7-x^3)^{1/3}$

so $y^2 = ((7-x^3)^{1/3})^2$

$$y^2 = (7-x^3)^{2/3}$$

Substitute into $y' = -\frac{x^2}{y^2}$

$$y' = -\frac{x^2}{y^2} = -\frac{x^2}{(7-x^3)^{2/3}}$$

this result matches result of [Example 1]

Diagram of what we did in these examples

Start with equation involving x, y

Solve for y

$y =$ function of x

ordinary derivative

$$y' = -\frac{x^2}{(7-x^3)^{2/3}}$$

implicit differentiation

$$y' = -\frac{x^2}{y^2}$$

Solved the original equation for y so we could replace the y^2 with an expression involving x.

Any time the equation can be solved for y in terms of x, these two processes will give the same result.

But the top process (as in [Example 1] is the clearest and most reliable.)

[Example 3] Suppose $x^2 + xy + y^2 = 9$

(note: this equation cannot be solved for y in terms of x.)

ⓐ find $\frac{dy}{dx}$ using Implicit Differentiation

Solution

Start with $x^2 + xy + y^2 = 9$ equation involving x, y

Step 1 $\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}(9)$

power function

$\frac{d}{dx}x^2 + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$

power rule   product!   product rule   nested function $(y)^2$ chain Rule

$2x + \left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right) + 2(y)\cdot y' = 0$

$2x + 1\cdot y + xy' + 2yy' = 0$

$2x + y + xy' + 2yy' = 0$ equation involving x, y, y'

Chain Rule Details

inner$(x) = y$

inner$'(x) = y'$

outer$(\ ) = (\ )^2$

outer$'(\ ) = 2(\ )$

Step 2 Solve for y'

$$2x + y + xy' + 2yy' = 0$$

$$xy' + 2yy' = -2x - y$$

$$y'(x + 2y) = -2x - y$$

divide

$$\boxed{y' = \frac{-2x - y}{x + 2y}}$$

(b) Find the equation for the line tangent to the ellipse

$$x^2 + xy + y^2 = 9$$

at the point $(-3, 3)$

Solution

We need to build the equation

$$(y - b) = m(x - a)$$

Where (a, b) are the (x, y) coordinates of the point of tangency, and m is the slope of the tangent line.

Clearly, $a = -3$ and $b = 3$, because we are given that the point of tangency is $(-3, 3)$

The slope m will be the value of y' when

$x = -3$ and $y = 3$.

That is

$$y' = \frac{-2x-y}{x+2y} = \frac{-2(-3)-3}{(-3)+2(3)} = \frac{6-3}{-3+6} = \frac{3}{3} = 1$$

Assembling the tangent line equation, we get

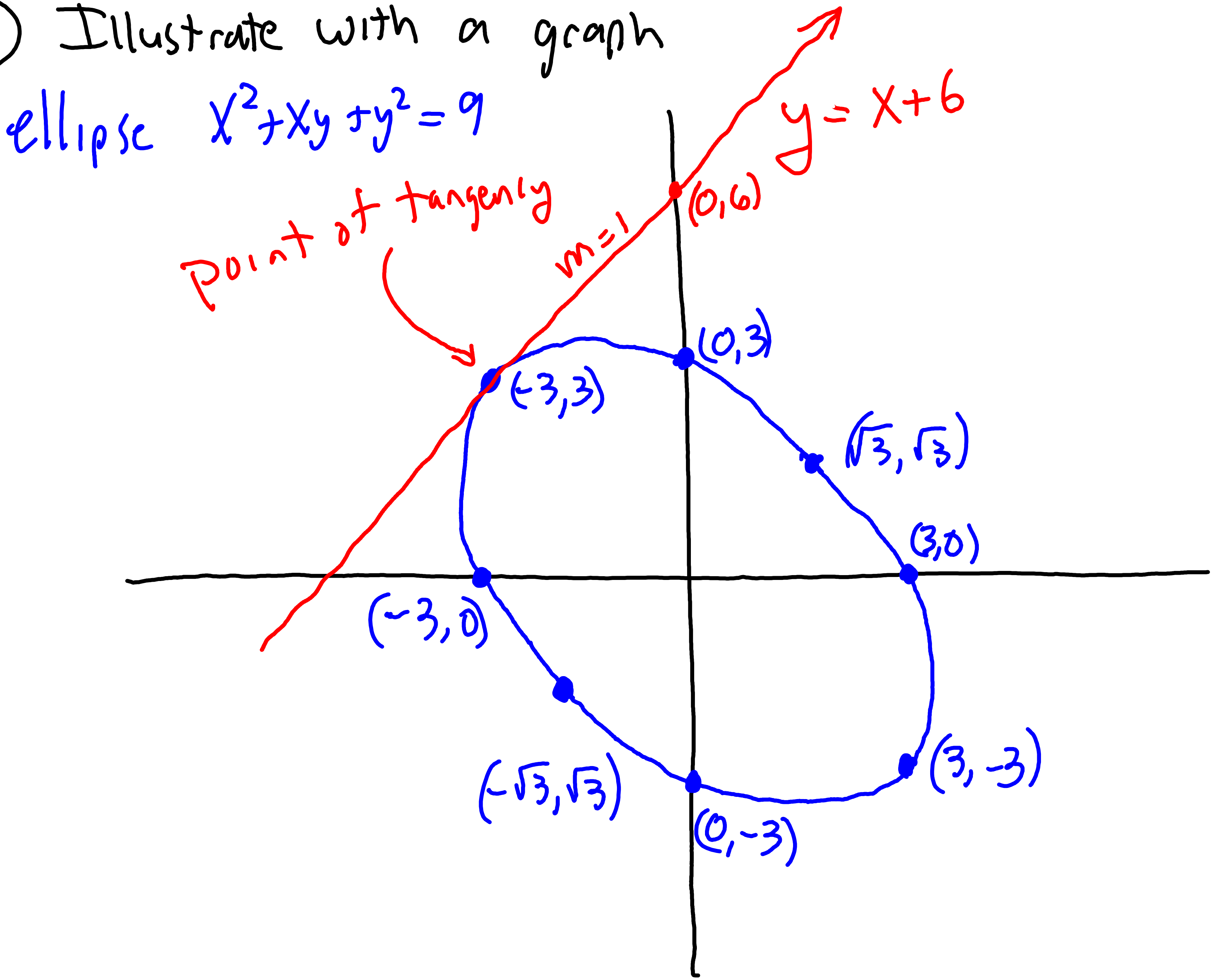
$$y-(3) = (1)(x-(-3))$$

$$y-3 = x+3$$

We put this into slope intercept form by solving for y

$$y = x+6$$

(c) Illustrate with a graph



End of Lecture