Pick up Handout

Sign In

Remember

Quiz Q5 Wednesday (Oct 11)

Holiday Friday

You can pick up graded Quiz Q4 tomorrow (Oct 10)
from the box outside my office, Morton 521
in the afternoon or evening

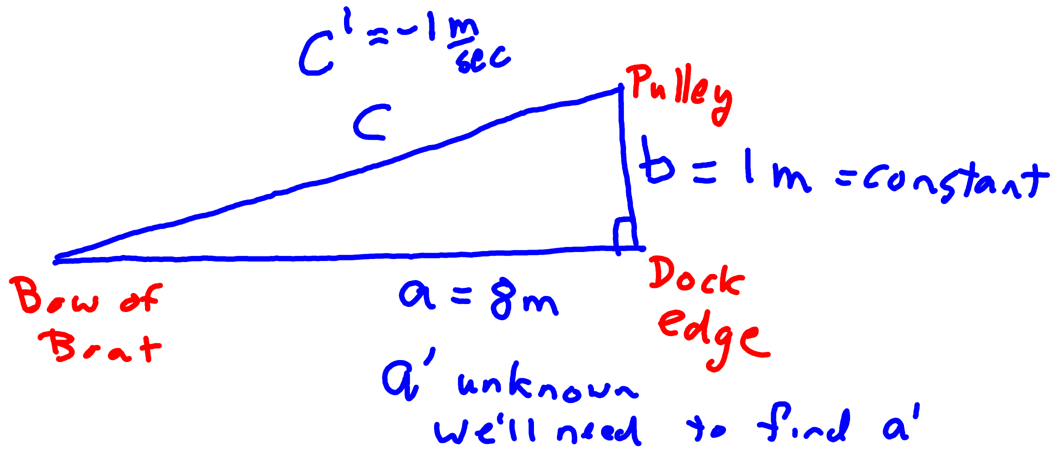
Meeting Part 1 More Related Rates Examples

[Example 1] (Involving Pythagorean Theorem) (exercise 2.7 #20)

Boat pulled towards a dock by a rope attached to bow of boat.
At the dock end, the rope goes over a pulley that is 1 m higher than bow of boat.
Rope is being pulled in at a rate of 1 m/sec.
How fast is boat approaching dock when it is 8 m from the dock?

Solution

Diagram + Variables



Equation Relating the variables

$$a^2 + b^2 = c^2$$ Pythagorean Theorem

Different w.r.t time t
(with respect to)

$$\frac{d}{dt}\left(a^2 + b^2\right) = \frac{d}{dt}c^2$$

$$\frac{d}{dt}a^2 + \frac{d}{dt}b^2 = \frac{d}{dt}c^2$$

a is changing
So a is a
function of t
So this is a
nested function $(a)^2$
Need to use Chain Rule

Chain Rule Details

inner(t) = a
inner'(t) = a'
outer( ) = $(\ )^2$
outer'( ) = 2( )

$$2(a)\cdot a' + 2(b)b' = 2(c)c'$$

$$2aa' + 2bb' = 2cc'$$

divide everything by 2

④

$$a a' + b b' = c c'$$

$b' = 0$ because b constant

$$a a' = c c'$$

Solve for a'

$$a' = \frac{c c'}{a}$$

Substitute in known values

$$= \frac{\sqrt{65} \cdot \cancel{m} \; (-1 \text{m/sec})}{8 \cancel{m}}$$

$$a' = -\frac{\sqrt{65}}{8} \frac{m}{sec}$$

The boat is approaching the dock at a rate of $\frac{\sqrt{65}}{8} \frac{m}{s}$

figure out c using Pythagorean Theorem

$$a^2 + b^2 = c^2$$

$$c = \sqrt{a^2 + b^2}$$

$$= \sqrt{(8m)^2 + (1m)^2}$$

$$= \sqrt{64 m^2 + 1 m^2}$$

$$= \sqrt{65 \, m^2}$$

$$= \sqrt{65} \cdot m$$

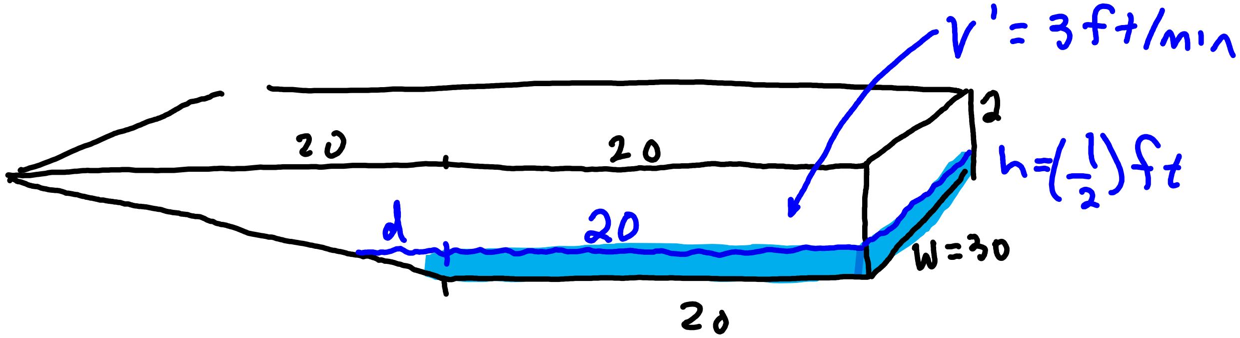
[Example 2] Involving basic Shapes (Similar to 2.7 #25, 26)

Swimming pool is 30ft wide, 40ft long with cross section as shown

Pool being filled at a rate of $3\,ft^3/min$.

How fast is the water level rising at instant when depth is $\left(\frac{1}{2}\right)$ft at deep end?

Solution

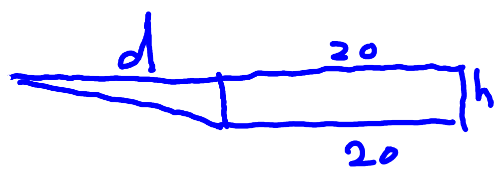


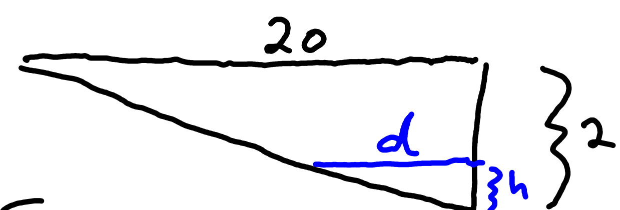
$h' =$ unknown

goal find h'

$V =$ area of side $\cdot$ width

We have to figure out d.

$=$  $\cdot\ 30$ft



$= \left(\frac{1}{2} h d + 20ft \cdot h\right) \cdot 30ft$

$= \left(\frac{1}{2} h \cdot 10h + 20ft \cdot h\right) \cdot 30ft$

Similar triangles

$\frac{d}{h} = \frac{20}{2} = 10$

$d = 10h$

$V = (5h^2 + 20ft \cdot h) \cdot 30ft$ equation relating variables h, V

Take Derivative of both sides $\frac{d}{dt}$ using Implicit Diff

Result after some steps.

$V' = (10h \cdot h' + 20ft \cdot h') \cdot 30ft$

Solve for h'

Result after some steps $h' = \dfrac{V'}{30ft(20ft + 10h)}$

Substitute in numbers: $V' = 3ft^3/min$ $h = \frac{1}{2}ft$

Result

$$h' = \frac{1}{250}\frac{ft}{min}$$

Meeting Part 2 Linear Approximations

Given function $f(x)$, the equation for line tangent to graph of f at $x=a$ is

$$y - f(a) = f'(a)(x-a)$$

In this equation, $a, f(a), f'(a)$ are all numbers.

Solve for y by adding $f(a)$ to both sides

$$y = f(a) + f'(a)(x-a)$$

This equation gives us y values on tangent line as a function of x.

Give this function a name and a symbol

The Linearization of $f(x)$ at a

$$L(x)$$

Linearizations and the Method for Finding a Linear Approximation
MATH 2301 (Barsamian)

Definition of the Linearization

Words: The ***linearization*** of $f(x)$

Meaning: The function $L(x)$ defined by the equation

$$L(x) = f(a) + f'(a)(x - a)$$

Graphical Significance: $L(x)$ describes the line that is tangent to the graph of $f(x)$ at $x = a$.

Method for Finding a Linear Approximation

Given: a function $f(x)$ and a hard x value called $\hat{x}$. (That is, it is not easy, or maybe even not possible, to compute $f(\hat{x})$ exactly by hand.)

Goal: Find an ***approximation*** for $f(\hat{x})$.

Steps:

- Identify the function $f(x)$
- Identify the hard x value, called $\hat{x}$.
- Identify an easy nearby x value, called a. That is, such that $f(a)$ is easy to compute.
- Build the ***linearization*** of $f(x)$ at a. That is, build the function
$$L(x) = f(a) + f'(a)(x - a)$$
- Use the ***linearization*** to compute the number $L(\hat{x})$. That is, compute
$$L(\hat{x}) = f(a) + f'(a)(\hat{x} - a)$$
(This should be an easy calculation.) This number $L(\hat{x})$ is the desired ***approximation*** for $f(\hat{x})$. It is called the ***linear approximation*** for $f(\hat{x})$

Example Let $f(x) = x^2$

Find the linearization of f at $x = 3$.

Solution

We need to build

$$L(x) = f(a) + f'(a)(x - a)$$

get parts

$a = 3$ ← x coord of P.O.T.

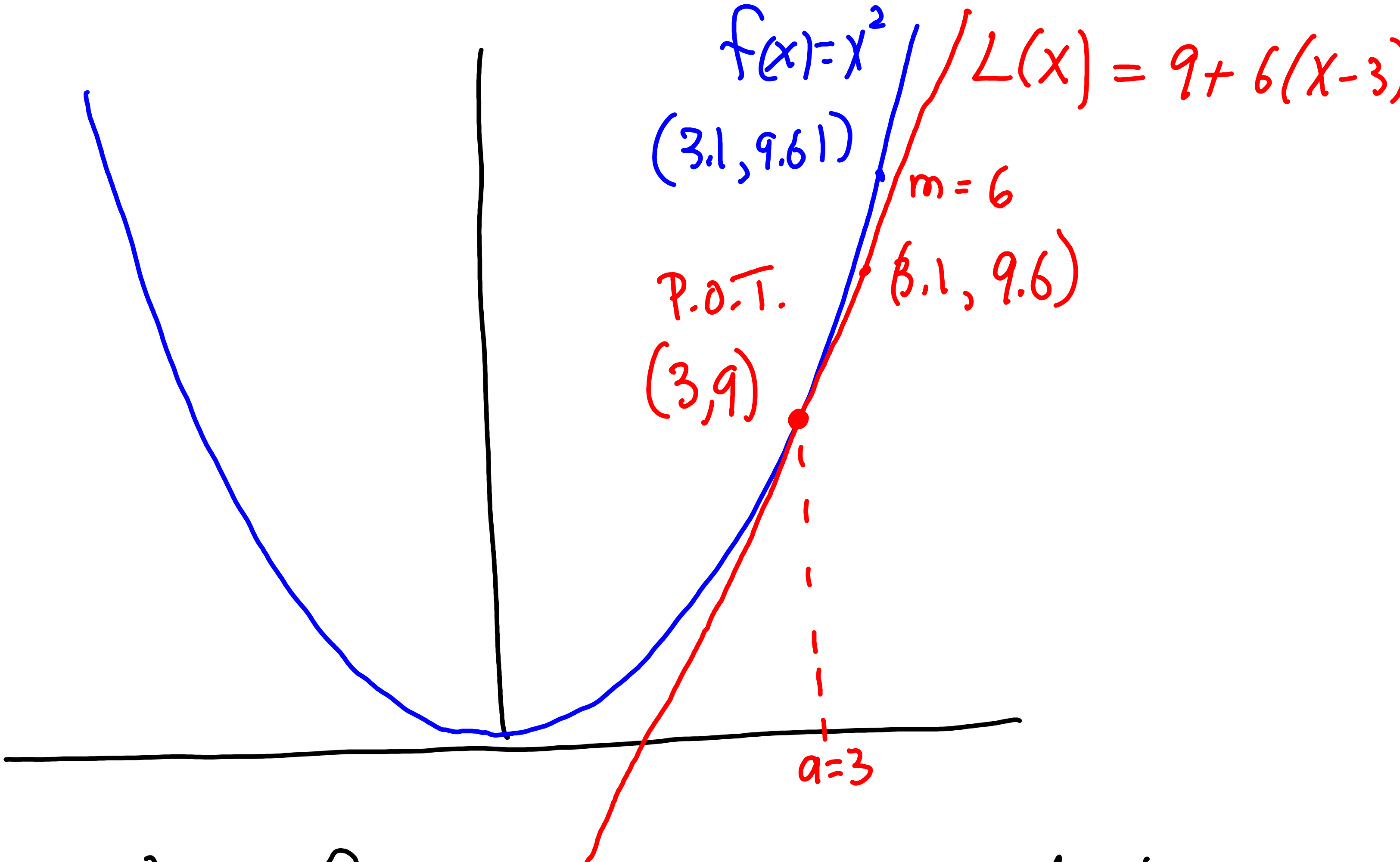
$f(a) = f(3) = (3)^2 = 9$ ← y coord of P.O.T

$f'(x) = 2x$

$f'(a) = f'(3) = 2(3) = 6$ ← slope m

assemble equation

$$L(x) = 9 + 6(x - 3)$$



Observe $L(3) = f(3) = 9$ because point of tangency

But $f(3.1) = (3.1)^2$ = harder computation ... $= 9.61$

$$L(3.1) = 9 + 6(3.1-3) = 9 + 6(.1) = 9 + .6 = 9.6$$

Observe: $L(3.1)$ is not the same as $f(3.1)$, but $L(3.1)$ was easier to compute than $f(3.1)$.

Because $f(3.1)$ is hard to compute, I like to think of $x = 3.1$ as a "hard" x value. I denote this by putting a little hat over the x, kind of like a <u>dunce cap</u>. So $\hat{x} = 3.1$

The number $L(3.1)$ is called the <u>Linear Approximation</u> of $f(3.1)$

That is, $L(\hat{x})$ is called the Linear Approximation of $f(\hat{x})$

In general, for a given function $f(x)$, and some hard x value, denoted $\hat{x}$, such that $f(\hat{x})$ would be hard to compute, we can take a similar approach: Build a Linearization, $L(x)$, and then use that linearization to find $L(\hat{x})$ using an easy computation. The number $L(\hat{x})$ is called the Linear Approximation of the number $f(\hat{x})$

This process of approximating is described in the second part of the handout that is included in these notes as page 8, above.

End of Lecture