Pick up your Graded Work

Sign In

Exam X2 this coming Friday, Oct 20

Common Missed Opportunity: Cancel Common Factors

Cancel Early, as soon as the common factors appear.

Otherwise, you will end up having to rewrite common factors on subsequent steps. This takes time, clutters your work, and increases chances of making a mistake.

In particular, **Cancel Before Multiplying!!!** (CBM)

Otherwise, your multiplication step is harder, and your simplifying will be harder, too!

The next six pages show samples of student work from Quiz Q5. In all six of these samples, the student did end up with the right answer for the calculation. But you can see that they made their calculations more cluttered and more difficult by not cancelling the common factor of 2 at the earliest opportunity. And they all had to eventually cancel the 2 in the simplifying, anyway. (Except for the one student who did not simplify and who lost a point for not simplifying!)

(**Not Shown**: There were many more Q5 papers with this same error. I only included the samples where the student did end up with the right answer.)

$$a^2 + b^2 = c^2$$

$$\frac{d}{dt}(a^2 + b^2) = \frac{d}{dt}c^2$$

$$a^2\frac{d}{dt} + b^2\frac{d}{dt} = c^2\frac{d}{dt}$$

$$2(a)a' + 2(b)b' = 2(c)c'$$

$$2(5)1 + 2(12)b' = 2(13)0$$

$$10 + 24b' = 0$$

$$\frac{24b'}{24} = -10$$

$$\boxed{b' = -\frac{10}{24}}$$

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$a^2 + b^2 = c^2$

a 5

$\frac{d}{dx} a^2 + \frac{d}{dx} b^2 = \frac{d}{dx} c^2$

$a' = 1$

Chain rule for all 3

example!

$\frac{d}{dx}(a)^2 = 2aa'$

$2aa' + 2bb' = 2cc'$

rearange

$2bb' = 2cc' - 2aa'$

Plug-in

$2(12)b' = 2(13)(0) - 2(5)(1\,ft/sec)$

$a = 5$
$c = 13$
$b = 12$
$a' = 1$
$c' = 0$

Solve for b'

$24b' = 0 - 10\ ft/sec$

$b' = -\frac{10}{24} ft/sec$

$\boxed{b' = -\frac{5}{12} ft/sec}$

Ladder is sliding down the wall at $\frac{5}{12}$ ft/sec

$\left[a^2 + b^2 = c^2\right] \frac{d}{dt}$

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$b^2 = c^2$

$b = \sqrt{16}$

$b = \sqrt{1}$

$b = 12$

chain rule

$2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} = 2c \cdot \frac{dc}{dt}$

we know c is a constant (13), so $\frac{dc}{dt} = 0$

plug in:

$2(5) \cdot 1 + 2(12) \frac{db}{dt} = 0$

$10 + 24 \frac{db}{dt} = 0$

$24 \frac{db}{dt} = -10$

$\frac{db}{dt} = -\frac{10}{24} = \boxed{-\frac{5}{12} \text{ ft/sec}}$

-25 $\quad -25$ $\quad b = 12$

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$$\left(\frac{d}{dt}\right) a^2 + b^2 = \left\{c^2\right\} \frac{d}{dt}$$

c is a constant so $c' = 0$

$$2aa' + 2bb' = 2cc'$$

$$2aa' + 2bb' = 0$$

$$\frac{2bb'}{2b} = \frac{-2aa'}{2b}$$

$$b' = \frac{-2aa'}{2b}$$

$$b' = \frac{-2(5ft)(-1\ ft/sec)}{2(12ft)} = \frac{-10\ ft^2/sec}{24\ ft} = -\frac{5}{12}\frac{ft}{sec}$$

The top of the ladder is sliding down the wall at the rate when the foot of the ladder is 5ft from the wall at a rate $\frac{5}{12}\frac{ft}{sec}$

The Quiz continues on back ➔

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$$a^2 + b^2 = c^2$$

$$\frac{d}{dx}(a^2) + \frac{d}{dx}(b^2) = \frac{d}{dx}(c^2)$$

$$2aa' + 2bb' = 2cc'$$

$$2bb' = 2cc' - 2aa'$$

$$b' = \frac{2cc'}{2b} - \frac{2aa'}{2b}$$

$$b' = \frac{2(13)(0)}{2(12)} - \frac{2(5)(1)}{2(12)}$$

$$b' = -\frac{2(5)}{2(12)}$$

$$b' = -\frac{10}{24}$$

$$\boxed{b' = -\frac{5}{12}\ \text{ft/sec}}$$

Meeting Part 1 Logarithmic Differentiation

Monday, we discussed Derivative Rules

Old Rule: Power Rule $\frac{d}{dx} x^n = n x^{n-1}$

power function

New Rule: Exponential Function Rule $\frac{d}{dx} b^{(x)} = b^{(x)} \ln(b)$

Special case $\frac{d}{dx} e^{(x)} = e^{(x)}$

New Rule: Logarithmic Function Rule $\frac{d}{dx} \ln(x) = \frac{1}{x}$ restricted to $x > 0$

$\frac{d}{dx} \ln(|x|) = \frac{1}{x}$ true for all $x \neq 0$

General case $\frac{d}{dx} \log_b(x) = \frac{1}{x \ln(b)}$ restricted to $x > 0$

We did not know how to find this derivative:

$$\frac{d}{dx} x^x = ??$$

Use Technique of Logarithmic Differentiation used for finding derivative of functions of the form

$$f(x) = g(x)^{h(x)}$$

To find $f'(x)$

(1) Take $\ln(\ )$ of both sides to obtain

$$\ln(f(x)) = \ln\left(g(x)^{h(x)}\right) = h(x)\ln(g(x))$$

$\ln(a^b) = b\ln(a)$

(2) Now find $\frac{d}{dx}(\ )$ of both sides to obtain

$$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(h(x)\ln(g(x)))$$

result will be an equation involving $f'(x)$

(3) Solve for $f'(x)$

[Example 1] $f(x) = x^x$. Find $f'(x)$

Solution use Logarithmic differentiation

$$f(x) = x^x$$

Step 1 $$\ln(f(x)) = \ln(x^x) = x\ln(x)$$

Step 2 $$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(x\ln(x))$$

nested function need chain rule!

Product rule!

Chain Rule Details
inner$(x) = f(x)$
inner$'(x) = f'(x)$
outer$(\ ) = \ln(\ )$
outer$'(\ ) = \frac{1}{(\ )}$

$$\frac{1}{(f(x))} \cdot f'(x) = \left(\frac{dx}{dx}\right)\ln(x) + x\left(\frac{d}{dx}\ln(x)\right)$$

$$= (1)\ln(x) + x\left(\frac{1}{x}\right)$$

$$= \ln(x) + 1$$

Step 3 multiply both sides by $f(x)$

$$f'(x) = f(x)(\ln(x)+1) = x^x(\ln(x)+1)$$

end of example

[Example 2] $f(x) = (\sin(x))^{\cos(x)}$. Find $f'(x)$

Solution: Use Logarithmic Differentiation

Step 1 $\quad \ln(f(x)) = \ln\left(\sin(x)^{\cos(x)}\right) = \cos(x)\cdot\ln(\sin(x))$

Step 2 $\quad \underbrace{\frac{d}{dx} f(x)}_{\text{chain rule}} = \underbrace{\frac{d}{dx}\Big(\cos(x)\cdot\ln(\sin(x))\Big)}_{\text{product rule}}$

$$\frac{1}{f(x)}\cdot f'(x) = \left(\frac{d}{dx}\cos(x)\right)\cdot\ln(\sin(x)) + \cos(x)\underbrace{\frac{d}{dx}\ln(\sin(x))}_{\text{chain}}$$

$$= (-\sin(x))\cdot\ln(\sin(x)) + \cos(x)\cdot\frac{1}{(\sin(x))}\cdot\cos(x)$$

$$= -\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}$$

Chain Rule Details

$\text{inner}(x) = \sin(x)$

$\text{inner}'(x) = \cos(x)$

$\text{outer}(\ ) = \ln(\ )$

$\text{outer}'(\ ) = \frac{1}{(\ )}$

Step 3 Solve for $f'(x)$ by multiplying both sides by $f(x)$

$$f'(x) = f(x)\cdot(\underbrace{\text{all that stuff}})$$

$$= \sin(x)^{\cos(x)}\cdot\left(-\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right)$$

end of example

Be careful to write correct, true equations.

often seen in corner of quiz & exam papers.

$$\cancel{\ln(x) = \frac{1}{x}}$$

not true!!

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \text{(restricted to } x>0\text{)}$$

Meeting Part 2 Exponential Growth + Decay (Section 3.4)

Very important derivative

$$\frac{d}{dx} e^{(kx)} = e^{(kx)} \cdot k$$

chain rule

$inner(x) = kx$

$inner'(x) = k$

$outer(\ ) = e^{(\ )}$

$outer'(\ ) = e^{(\ )}$

$$\boxed{\frac{d}{dx} e^{(kx)} = ke^{(kx)}}$$

More generally, if $f(x) = c \cdot e^{(kx)}$,

constant

Then

$$\frac{d}{dx} f(x) = \frac{d}{dx} c e^{(kx)} = c \cdot \frac{d e^{(kx)}}{dx} = c \cdot k e^{(kx)} = k \cdot c e^{(kx)} = k \cdot f(x)$$

$$\frac{d}{dx} f(x) = k \cdot f(x)$$

instantaneous rate of change of $f(x)$ $= k \cdot f(x)$

instantaneous rate of change of $f(x)$ is proportional to $f(x)$

Big Fact

The only functions that satisfy the equation

$$\frac{d}{dx} f(x) = k \cdot f(x)$$

Differential Equation
(equation involving derivatives)

are functions of the form

$$f(x) = c e^{(kx)}$$

When the variable is t for time, the function form

$$f(t) = C e^{(kt)}$$

describes "exponential growth" (when k is positive)

or "exponential decay" (when k is negative)

Radioactive Decay

Fact: Radioactive materials obey "exponential decay"

$$m(t) = C e^{(kt)}$$

Observe: when time $t=0$

$$m(0) = c\, e^{(k(0))} = c e^{0} = c \cdot 1 = c$$

So c is the "initial value". (the value at time $t=0$)

So instead of c, sometimes other symbols are used

$$m(t) = \underline{c}\, e^{(kt)} = \underline{m_0}\, e^{(kt)} = \underline{m(0)}\, e^{(kt)}$$

different ways of denoting the initial value

[Example 3] (3.4#9)

Cesium-137 has a half-life of 30 years.

Suppose we have a 100-mg sample

(a) Write a formula for the mass as a function of time.

Solution Radioactive decay

$$m(t) = m_0 e^{(kt)} = 100e^{kt} \text{ mg}$$

need to figure out value of k. Use fact of the half life:

$$m(30) = \frac{1}{2} m_0$$

mass at 30 years $= \frac{1}{2}$ starting mass turn this around

$$\frac{1}{2} m_0 = m(30) = m_0 e^{(k \cdot 30)}$$

Solve for k

Divide by m_0

$$\frac{1}{2} = e^{(k \cdot 30)}$$

take ln( ) of both sides

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{k \cdot 30}\right) = k \cdot 30$$

$\ln(e^a) = a$

$$k = \frac{\ln(1/2)}{30}$$

a negative number.

Put that into formula for $m(t)$

$$m(t) = 100e^{(k \cdot t)} = 100e^{\left(\frac{\ln(1/2)}{30} \cdot t\right)}$$

We can simplify this

$$= 100e^{\left(\ln 1/2 \cdot \frac{t}{30}\right)}$$

$$= 100\left(e^{\ln(1/2)}\right)^{t/30}$$

$$m(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{30}}$$

Simpler form

end of example

end of lecture