MATH 2301 (Barsamian) Lecture #24 (Mon Oct 30, 2023)

Today: Section 4.3

Tomorrow Recitation: Sections 4.2 & 4.3

Wednesday: Section 4.4

Friday: Quiz Q7 Will need to include 4.2, 4.3, 4.4

Section 4.3 Derivatives and the Shapes of Graphs

Definition of Increasing (Decreasing) Function (From p.7, in Chapter 1)

words: f is increasing (decreasing) on the interval I.

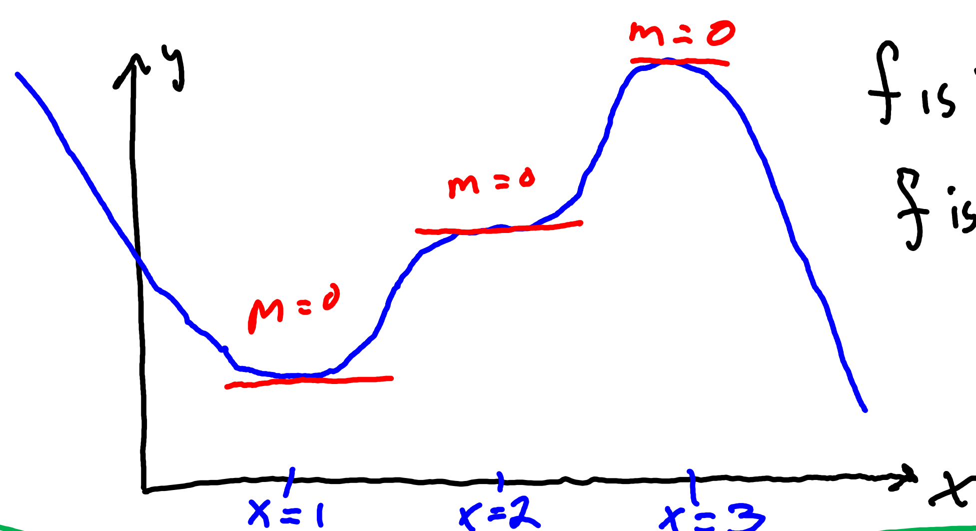
meaning: If $x_1, x_2 \in I$ and $x_1 < x_2$

are elements of

then $f(x_1) < f(x_2)$ $\quad (f(x_1) > f(x_2))$

I could be
(a, b)
$[a, b]$
$[a, b)$
$[a, \infty)$

graphical significance:
If you move from left to right in interval I,
the y values go up (y values go down)



f is increasing on the interval $[1, 3]$

f is decreasing on the intervals $(-\infty, 1]$ and $[3, \infty)$

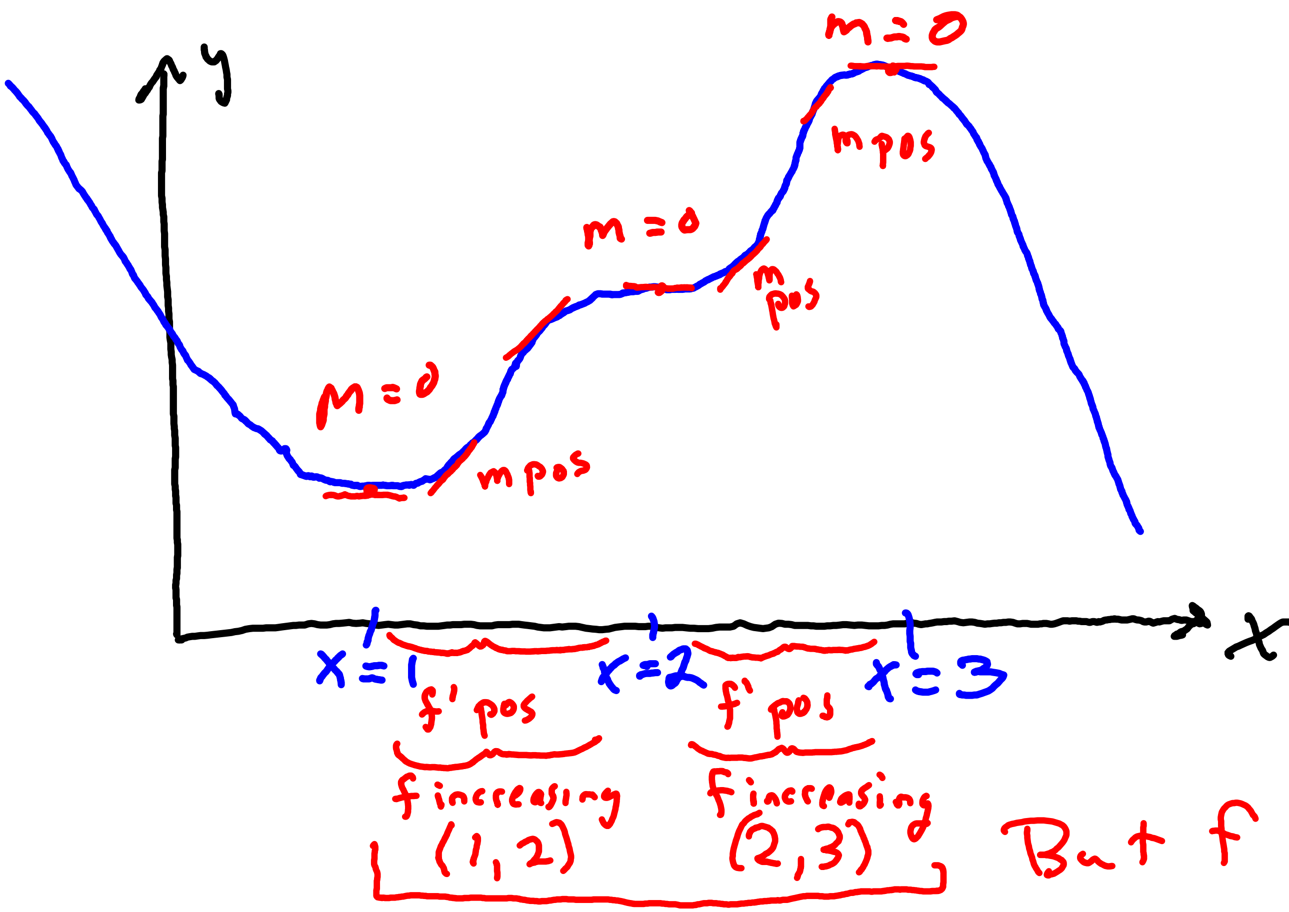
Important Consequence of Sign of $f'(x)$

Sign behavior of $f'(x)$ on an interval $\xrightarrow[\text{about}]{\text{tells us}}$ increasing/decreasing behavior of f on that interval

f' is pos on whole interval $\longrightarrow$ f increasing on whole interval

f' is neg on whole interval $\longrightarrow$ f decreasing on whole interval

f' is zero on whole interval $\longrightarrow$ f constant on whole interval



But f actually increasing on [1, 3]

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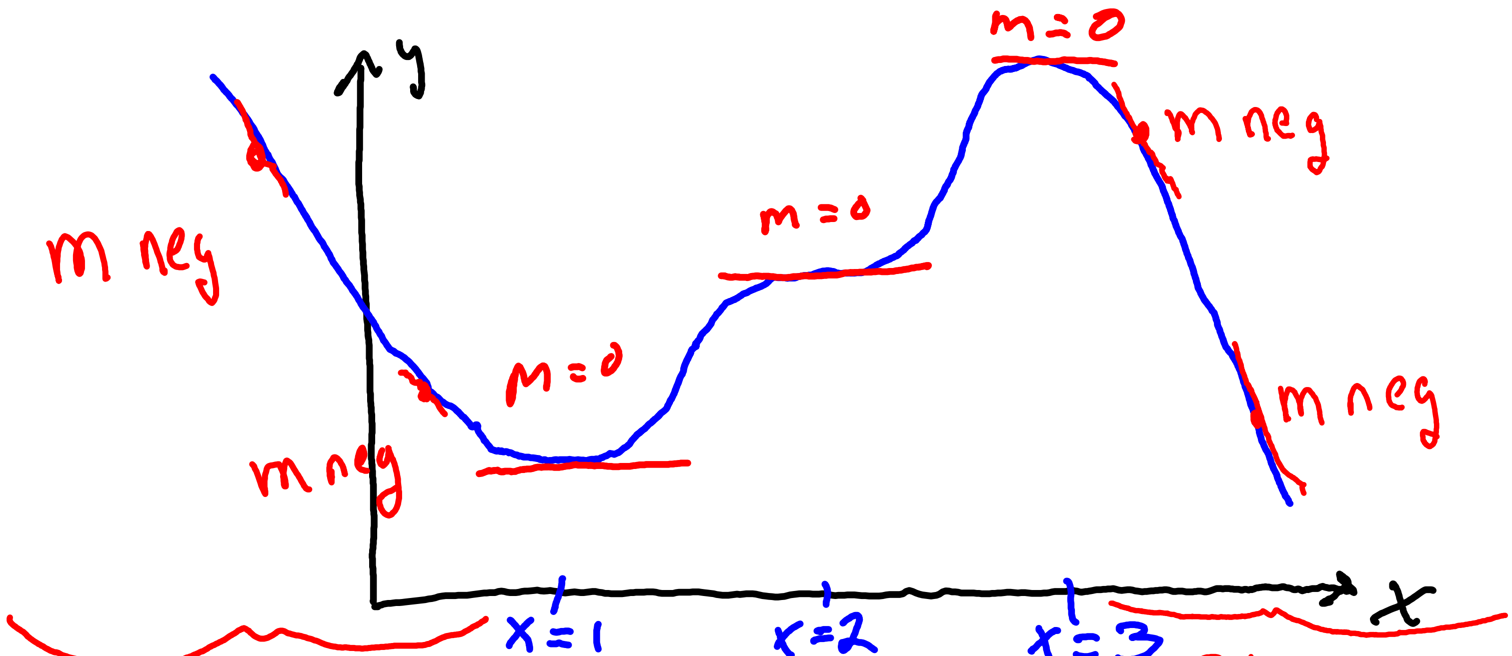


f′ neg on $(-\infty, 1)$

f decreasing on $(-\infty, 1)$

But actually f is decreasing on all of $(-\infty, 1]$

f′ is neg on $(3, \infty)$ tells us that f is definitely decreasing on $(3, \infty)$

But f is actually decreasing on all of $[3, \infty)$

<u>The First Derivative Test for Local Extrema</u>

Four Part Test

Test 1: $f'(c)=0$ or $f'(c)$ DNE

Test 2: $f(c)$ exists

If f, c pass tests 1, 2, we say

Test 3: f continuous at $x=c$

Test 4: f' changes sign at $x=c$

If f, c pass <u>all</u> <u>four</u> <u>tests</u> then $x=c$ is the <u>location</u> of a local max or local min

In this case, the y value $y=f(c)$ is the <u>local max value</u> or <u>local min value</u>.

[Example 1] Let $f(x) = x^4 + 6x^2 + 27$

(a) Find all intervals where f is increasing or decreasing

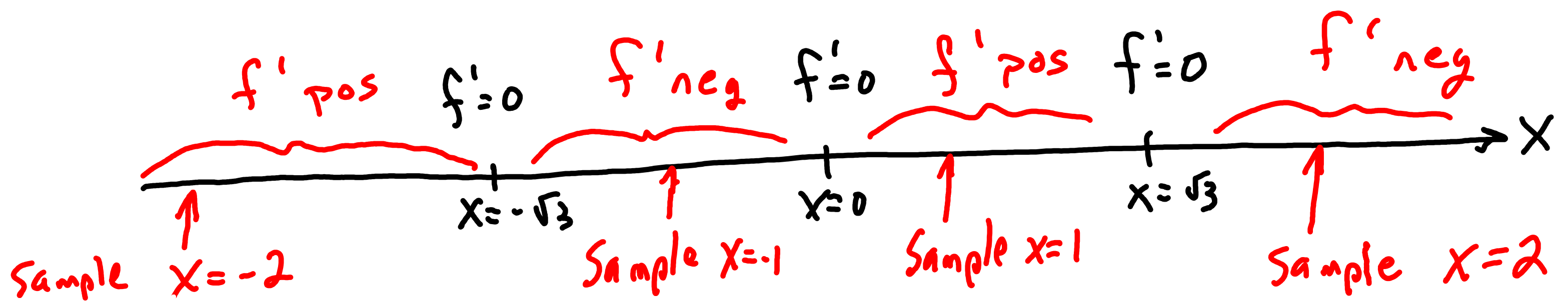
<u>Solution</u> Strategy: Study sign behavior of $f'(x)$

$$f'(x) = 4x^3 - 12x = -4x(x^2 - 3) =$$

$$= -4x(x+\sqrt{3})(x-\sqrt{3}) = -4(x+\sqrt{3})(x)(x-\sqrt{3})$$

roots: $x = -\sqrt{3}$ $\quad x = 0 \quad$ $x = \sqrt{3}$

Sign chart for $f'(x) = -4(x+\sqrt{3})\,x\,(x-\sqrt{3})$



$$f'(-2) = \underbrace{-4}_{(-)}\underbrace{((-2)+\sqrt{3})}_{(-)}\underbrace{(-2)}_{(-)}\underbrace{((-2)-\sqrt{3})}_{(-)} = \text{pos}$$

$$f'(-1) = \underbrace{-4}_{(-)}\underbrace{((-1)+\sqrt{3})}_{(+)}\underbrace{(-1)}_{(-)}\underbrace{((-1)-\sqrt{3})}_{(-)} = \text{neg}$$

$$f'(1) = \underbrace{-4}_{(-)}\underbrace{((1)+\sqrt{3})}_{(+)}\underbrace{(1)}_{(+)}\underbrace{((1)-\sqrt{3})}_{(-)} = \text{pos}$$

$$f'(2) = \underbrace{-4}_{(-)}\underbrace{((2)+\sqrt{3})}_{(+)}\underbrace{(2)}_{(+)}\underbrace{(2-\sqrt{3})}_{(+)} = \text{neg}$$

Conclusion

f increasing on intervals $(-\infty, -\sqrt{3}]$ and $[0, 3]$

because $f'(x)$ is pos or zero.

f decreasing on intervals $[-\sqrt{3}, 0]$ and $[0, \infty)$

because f' is neg or zero.

(b) Find x coordinates where f has local max or local min

Solution

By First Derivative Test,

f has local max at $x = -\sqrt{3}$ because f' changes $+, 0, -$

f has local min at $x = 0$ because f' changes $-, 0, +$

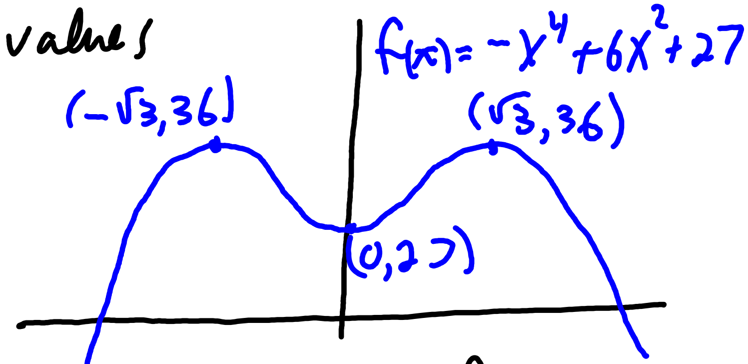
f has local max at $x = \sqrt{3}$ because f changes $+, 0, -$

(c) Find the local max values and local min values

Solution $y_{max} = f(-\sqrt{3}) = \cdots = 36$

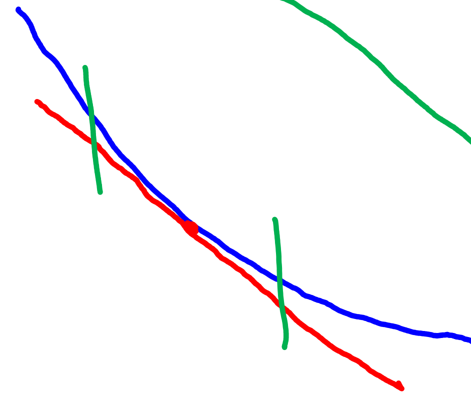
$y_{min} = f(0) = \cdots = 27$

$y_{max} = f(\sqrt{3}) = \cdots = 36$



End of example

Concavity and the Second derivative

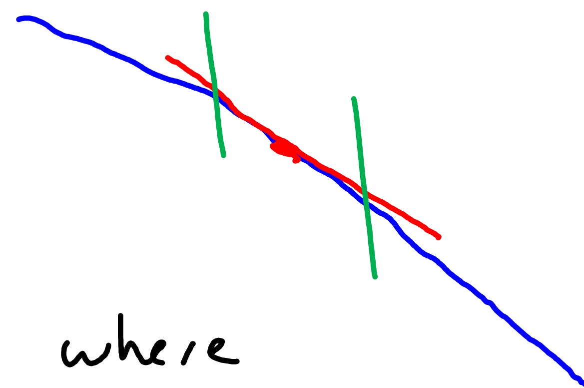


Definition of concave up on an interval

Words: f is concave up on an interval

meaning: At every x in the interval, there is a tangent line, and for x values near the point of tangency, the graph stays above the tangent line

Analogous Definition for concave down



A point of inflection is a point on graph where

- f is continuous
- Concavity changes from up to down or down to up

Important Relationship

Sign behavior of f'' $\xrightarrow[\text{about}]{\text{tells us}}$ concavity behavior of f

f'' is positive on a whole interval $\longrightarrow$ f concave up on whole interval

f'' is negative on a whole interval $\longrightarrow$ f concave down on whole

[Example 2] Return to $f(x) = -x^4 + 6x^2 + 27$

ⓐ Find intervals where f is concave up or concave.

Solution Strategy: Study sign behavior of f'' to answer question.

$f'(x) = -4x^3 + 12x$

$f''(x) = -12x^2 + 12 = -12(x^2 - 1) = -12(x+1)(x-1)$

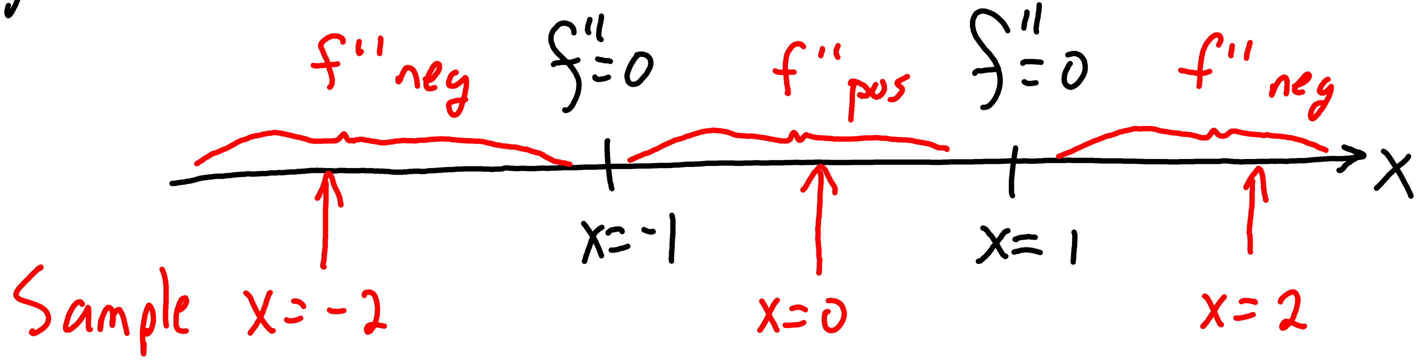
Set $f''(x) = 0$ and solve for x

$$0 = -12(x+1)(x-1)$$

Solutions are $x = -1$, $x = 1$

Make sign chart for $f''(x) = -12(x+1)(x-1)$

Sign of $f''(x)$



$f''(-2) = \underset{(-)}{-12}\underset{(-)}{((-2)+1)}\underset{(-)}{((-2)-1)} = \text{neg}$

$f''(0) = \underset{(-)}{-12}\underset{(+)}{((0)+1)}\underset{(-)}{((0)-1)} = \text{pos}$

$f''(2) = \underset{(-)}{-12}\underset{(+)}{((2)+1)}\cdot\underset{(+)}{((2)-1)} = \text{neg}$

f concave up on interval $[-1,1]$ because f'' is pos or zero there

f concave down on intervals $(-\infty,-1)$ and $(1,\infty)$ because f'' is neg or zero

(b) Find x coordinates of all inflection points on graph of $f(x)$

Solution

f has inflection points at $x = -1$ and $x = 1$ because f is continuous and f'' changes sign

(c) Find y coordinates of all inflection points on graph of $f(x)$

Solution $\quad f(x) = -x^4 + 6x^2 + 27$

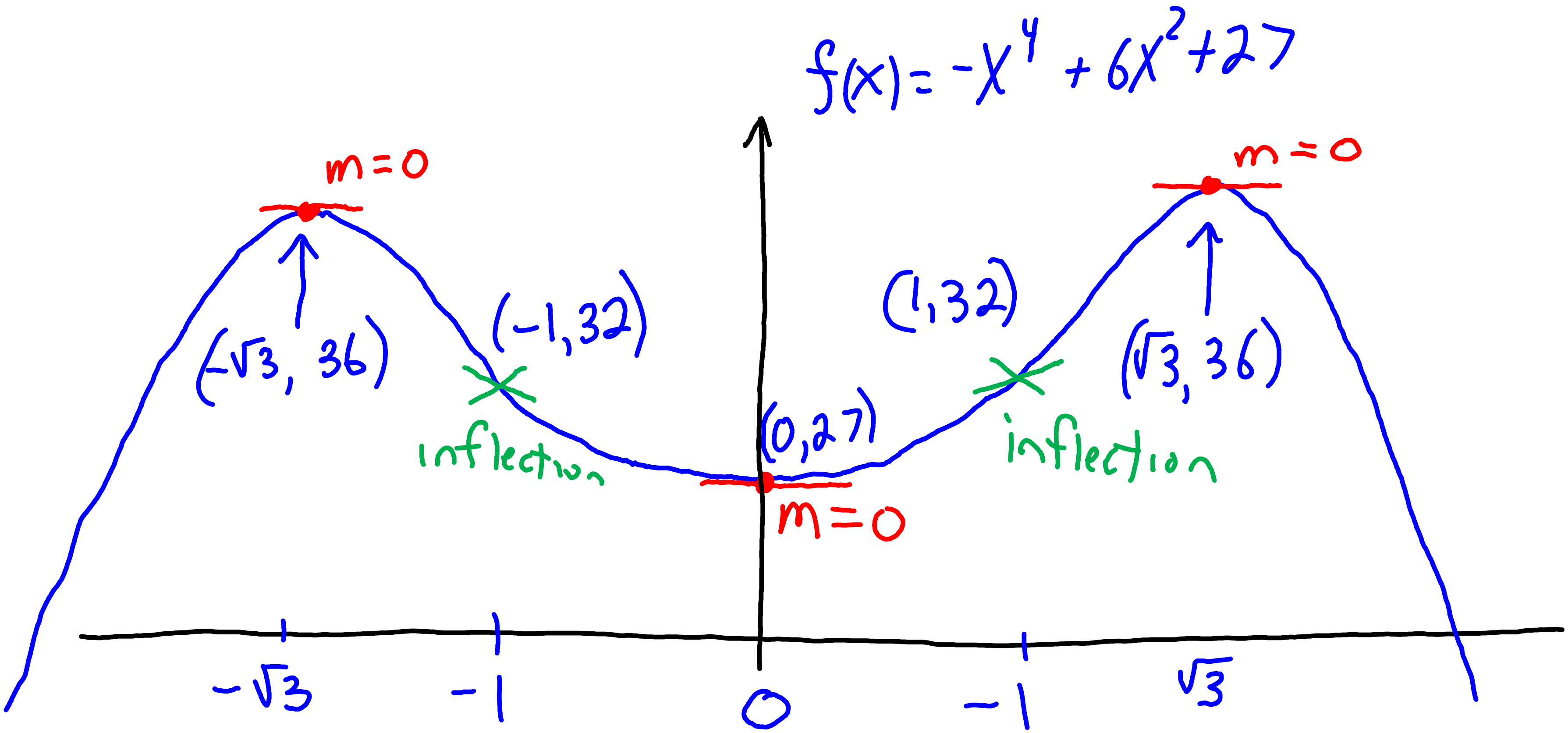
So $f(-1) = -(-1)^4 + 6(-1)^2 + 27 = -1 + 6 + 27 = 32$

$f(1) = -(1)^4 + 6(1)^2 + 27 = -1 + 6 + 27 = 32$

So the inflection points are at

$(x, y) = (-1, 32)$ and $(x, y) = (1, 32)$

d) Draw a graph of $f(x)$, showing the results of steps a, b, c, along with the results of [Example 1]



end of example

end of lecture