MATH 2301 (Barsamian) Lecture #25 (Wed Nov 1, 2023)

Pick Up Graded Work

Pick up Handout: Graphing Strategy

Sign In

Today: Section 4.4 Curve Sketching

Friday Quiz Q7 Will cover Sections 4.2, 4.3, 4.4

Section 4.4 Curve Sketching

Book has three excellent examples

[Example 1] Sketch $y = \frac{2x^2}{x^2-1}$

[Example 2] Sketch $y = xe^{(x)}$

[Example 3] Sketch $y = \frac{\cos(x)}{2+\sin(x)}$

Be sure to study those.

In particular, notice [Example 2] involves a function very similar to Quiz Q6 problem[1] $f(x) = xe^{(-2x)}$ or $xe^{(-3x)}$
your job was to find the critical numbers of $f(x)$. *

Notice that book's [Example 2] will of course involve finding critical numbers of $y = xe^{(x)}$

many forgot the 1

Many of you had difficulty factoring $f'(x) = e^{-2x} - 2xe^{(-2x)} = e^{-2x}(1-2x)$

[Example] Use Graphing Strategy to sketch a graph of

$$f(x) = \frac{1}{x^2+25}$$

Step 1

y intercept: Set $x=0$, find y $y = f(0) = \frac{1}{(0)^2+25} = \frac{1}{25}$. So $(0, \frac{1}{25})$ on graph.

x intercepts. Set $y=0$, find x. $0 = \frac{1}{x^2+25}$. No solutions! No x intercepts.

(Remember: fraction $\frac{a}{b} = 0$ only when numerator $a=0$ and denominator $b \neq 0$)

End Behavior

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2+25} = 0$$

x getting huge, positive

denominator getting really huge, positive

So the ratio will be positive number very close to 0.

So the limit is

So graph has horiz asymptote on right with line equation $y=0$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^2+25} = 0$$

x getting huge and negative

denominator huge, positive

So ratio will be positive number very close to 0

So graph has horiz asymptote on left with line equation $y=0$.

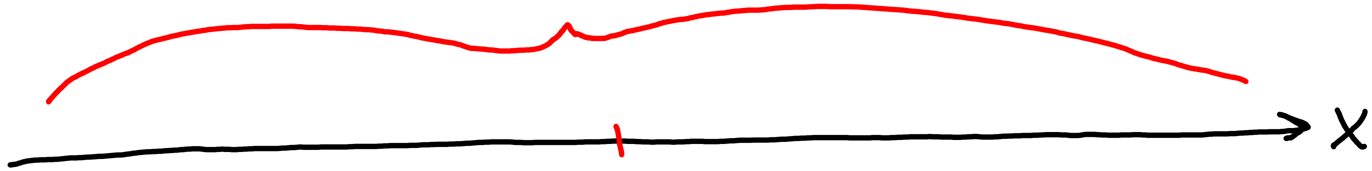
Vertical Asymptotes? <u>None</u> because the domain of $f(x) = \frac{1}{x^2+25}$ is all real numbers. There are no x values that cause denominator $= 0$

Continuity? Where is $f(x) = \frac{1}{x^2+25}$ continuous?

$f(x)$ is rational function. These are continuous <u>everywhere</u> except where denominator $= 0$. Our function's denominator is never zero. So $f(x)$ is always continuous

SignChart for $f(x) = \frac{1}{x^2+25}$

f(x) is always positive



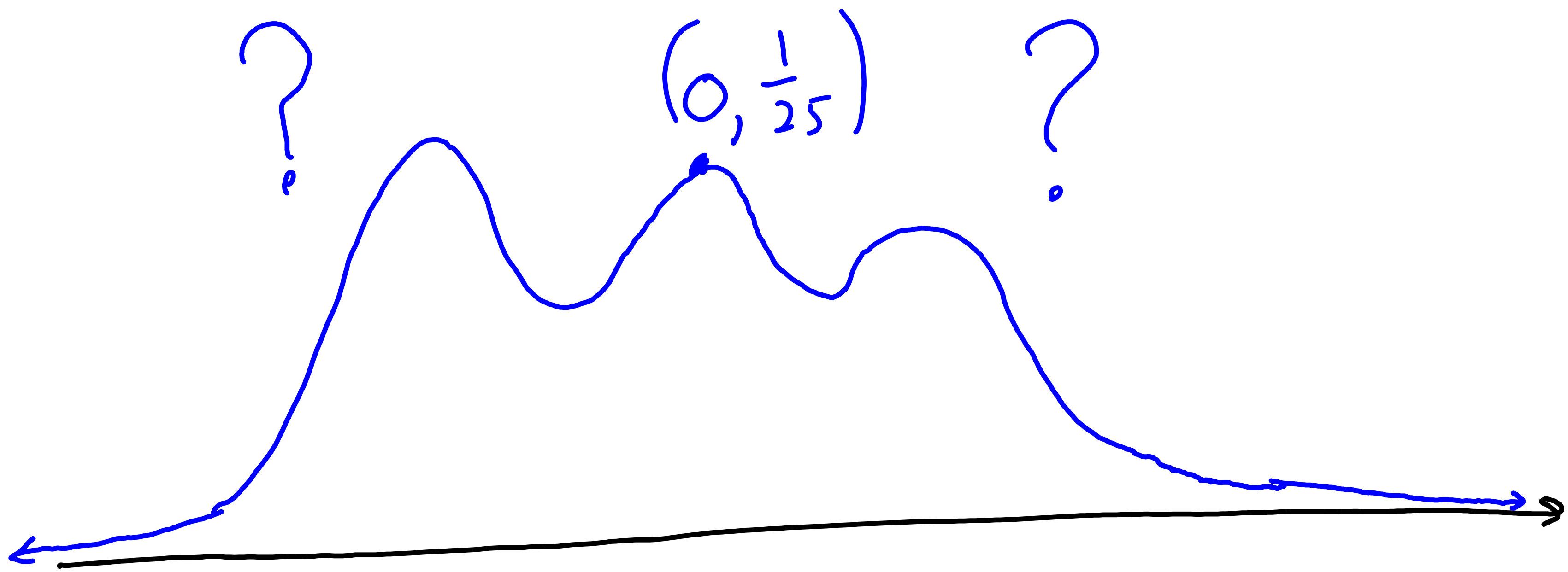
Sample $x=0$

no x values that cause $f(x)=0$

$$f(0) = \frac{1}{0^2+25} = \frac{1}{25} = \text{pos}$$

Preliminary Graph of $f(x) = \dfrac{1}{x^2+25}$ just based on Step 1

5



Step 2 Analyze $f'(x)$

$$f'(x) = \frac{d}{dx}\frac{1}{x^2+25} = \frac{\left(\frac{d}{dx}1\right)(x^2+25) - (1)\left(\frac{d}{dx}x^2+25\right)}{(x^2+25)^2} =$$

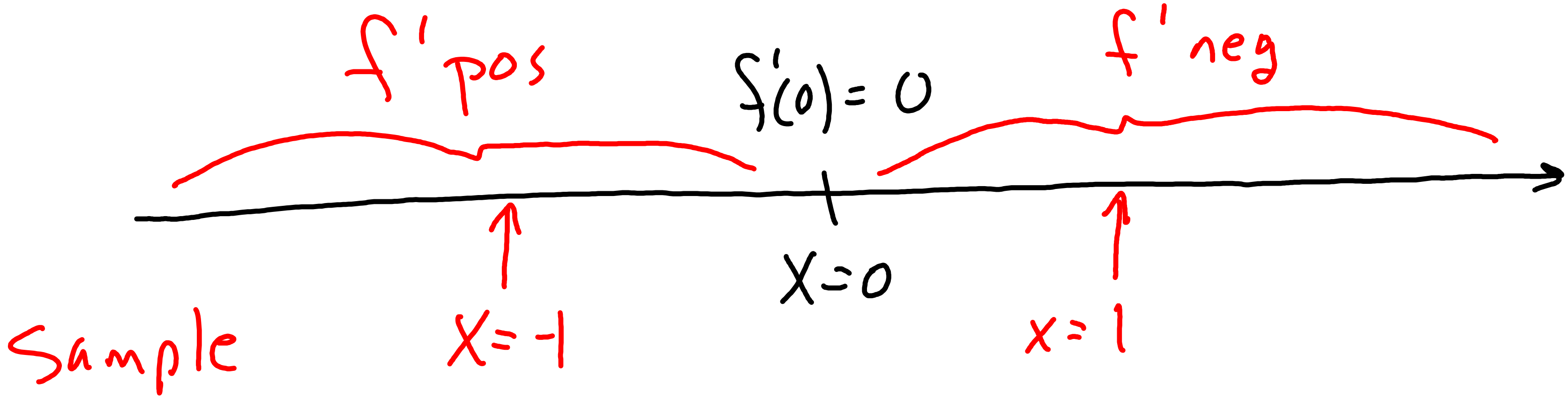
$$= \frac{\cancel{(0)(x^2+25)}\xrightarrow{} 0 - (1)(2x)}{(x^2+25)^2} = -\frac{2x}{(x^2+25)^2}$$

Are there any x values where $f'(x)$ is undefined? None because denominator will never be zero

Are there any x values that cause $f'(x) = 0$?

yes: $x=0$ because numerator $=0$ and denominator $\neq 0$

Sign chart for $f'(x) = \dfrac{-2x}{(x^2+25)^2}$



$$f'(1) = -\frac{2(1)}{((1)^2+25)^2} = -\frac{+}{+} = \text{neg}$$

$$f'(-1) = -\frac{2(-1)}{((-1)^2+25)^2} = -\frac{-}{+} = \text{pos}$$

f increasing on $(-\infty, 0]$ because f' is pos or zero there

f decreasing on $[0, \infty)$ because f' is neg or zero there.

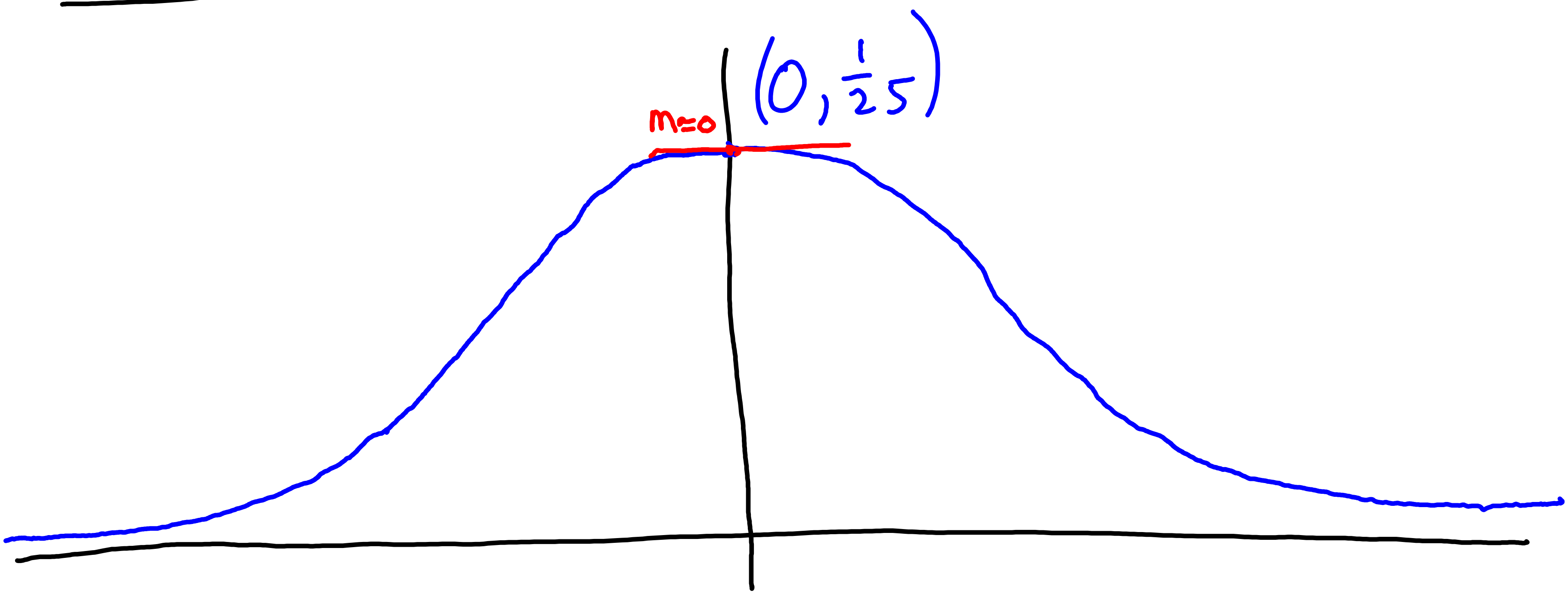
8

We know $x=0$ is a critical number for $f(x)$
because $f'(0)=0$ and $f(0)$ exists

By First Derivative Test tells us local max at $x=0$

Local max value is $y_{max} = f(0) = \frac{1}{25} = 0.04$

Preliminary Graph after Step 2



Step 3 Analyze $f''(x)$

9

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{-2x}{(x^2+25)^2}\right) = \frac{\left(\frac{d}{dx}-2x\right)(x^2+25)^2 - (-2x)\left(\frac{d}{dx}\overbrace{(x^2+25)^2}\right)}{\left((x^2+25)^2\right)^2} =$$

$$= \frac{(-2)(x^2+25)^2 + 2x\left(\overbrace{2(x^2+25)'\cdot(2x)}^{\text{chain rule}}\right)}{(x^2+25)^4}$$

$$= \frac{(-2)(x^2+25)^2 + 8x^2(x^2+25)}{(x^2+25)^4}$$

factor out common factor of x^2+25 on top

$$= \frac{\left[(-2)(x^2+25) + 8x^2\right](x^2+25)}{(x^2+25)^4}$$

Since $x^2+25 \neq 0$, we can cancel $\frac{x^2+25}{x^2+25}$

$$= \frac{\left[(-2)(x^2+25) + 8x^2\right]}{(x^2+25)^3} = \frac{-2x^2-50+8x^2}{(x^2+25)^3}$$

$$f''(x) = \frac{6x^2-50}{(x^2+25)^3}$$

Notice that the denominator of $f''(x)$ will never be zero. So there are no values of x that will cause $f''(x)$ to be undefined. To find values of x that will cause $f''(x)=0$, we only have to look for x that will cause the numerator to be zero.

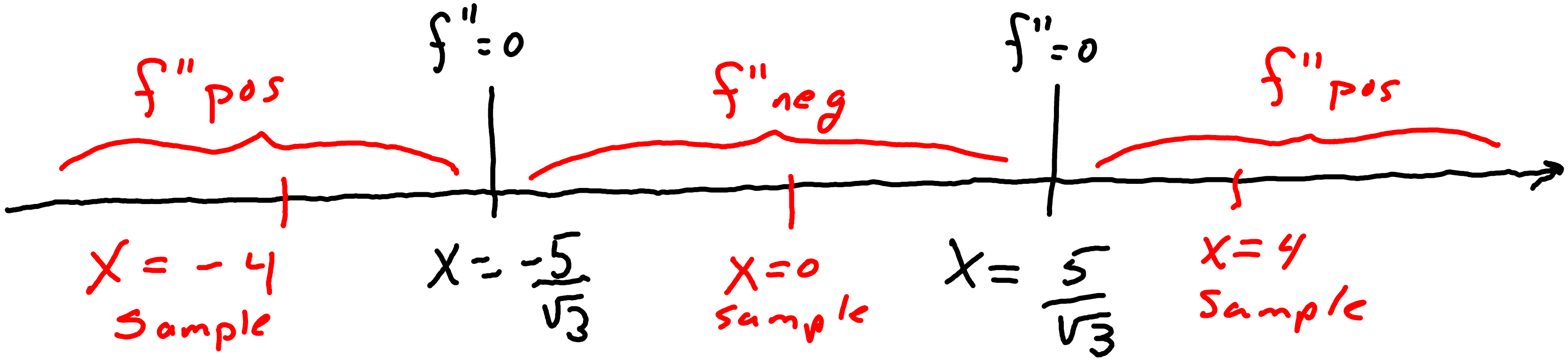
$$6x^2-50 = 0$$

$$6x^2 = 50$$

$$x^2 = \frac{50}{6} = \frac{25}{3}$$

$$x = -\frac{5}{\sqrt{3}} \approx -3 \qquad x = \frac{5}{\sqrt{3}} \approx 3$$

Sign chart for $f''(x) = \dfrac{6x^2-50}{(x^2+25)^3}$



$$f''(-4) = \frac{6(-4)^2-50}{((-4)^2+25)^3} = \frac{6(16)-50}{(16+25)^3} = \frac{96-50}{41^3} = \frac{+}{+} = \text{Pos}$$

$$f''(0) = \frac{6(0)^2-50}{(0^2+25)^3} = \frac{-50}{25^3} = \frac{-}{+} = \text{neg}$$

$$f''(4) = \frac{6(4)^2-50}{(4^2+25)^3} = \frac{96-50}{41^3} = \frac{+}{+} = \text{Pos}$$

f concave up on $\left(-\infty, -\frac{5}{\sqrt{3}}\right]$ and $\left[\frac{5}{\sqrt{3}}, \infty\right)$ because f'' is positive or zero there.

f concave down on $\left[-\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right]$ because f'' is negative or zero there.

f has points of inflection at $x = -\frac{5}{\sqrt{3}}$ and $x = \frac{5}{\sqrt{3}}$ because f is continuous there and the concavity changes from up to down or from down to up.

The y coordinates of the points of inflection are

x	$f(x) = \frac{1}{x^2+25}$
$-\frac{5}{\sqrt{3}}$	$f\left(\frac{-5}{\sqrt{3}}\right) = \frac{1}{\left(\frac{-5}{\sqrt{3}}\right)^2+25} = \frac{1}{\frac{25}{3}+25} = \frac{1}{\frac{25}{3}+\frac{75}{3}} = \frac{1}{\frac{100}{3}} = \frac{3}{100} = 0.03$
$\frac{5}{\sqrt{3}}$	$f\left(\frac{5}{\sqrt{3}}\right) = \frac{1}{\left(\frac{5}{\sqrt{3}}\right)^2+25} = \cdots = \frac{3}{100}$

So the two inflection points are at

$$(x,y) = \left(\frac{-5}{\sqrt{3}}, 0.03\right) \quad \text{and} \quad (x,y) = \left(\frac{5}{\sqrt{3}}, 0.03\right)$$

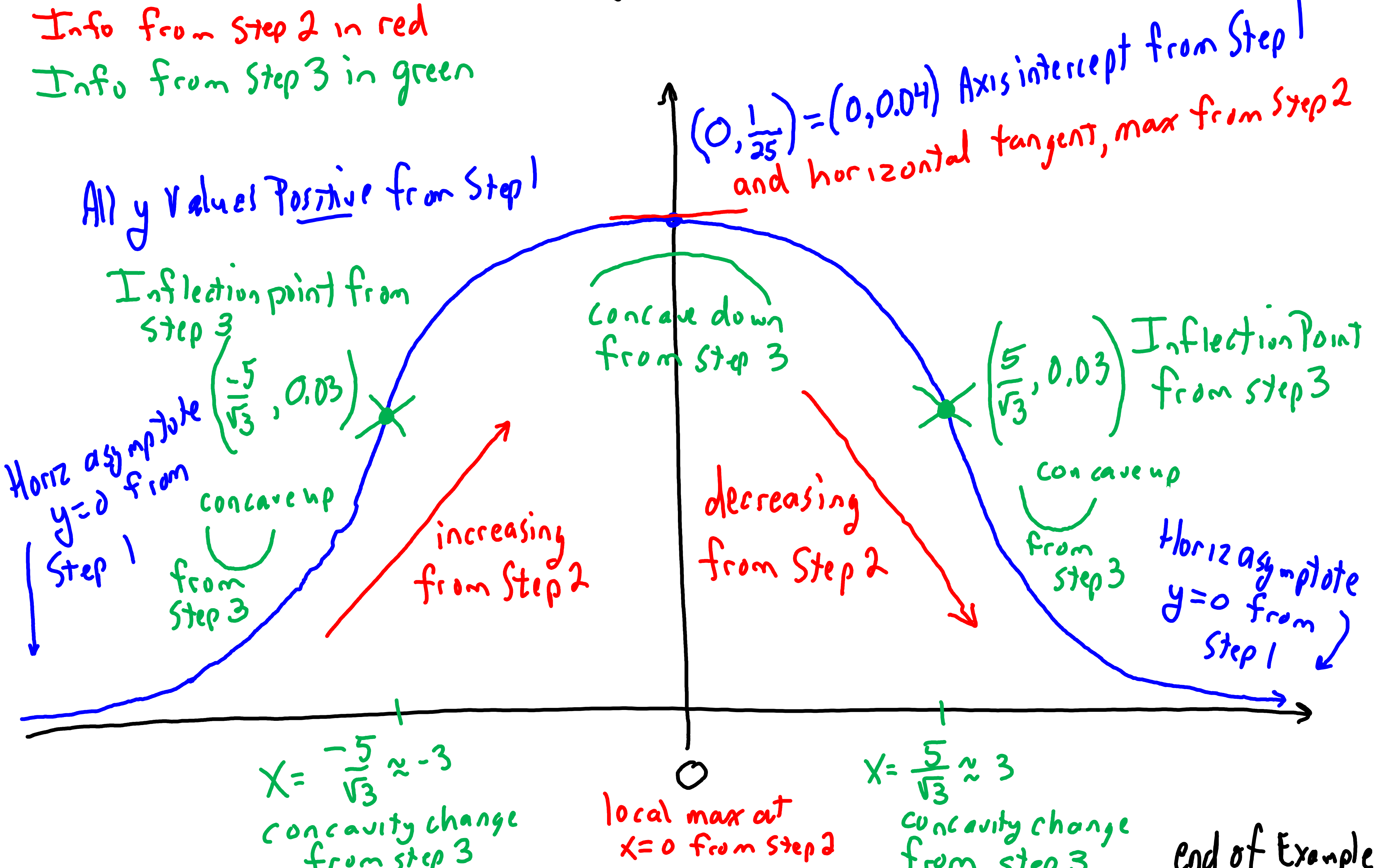
Step 4 Put together info from Steps 1, 2, 3 and sketch graph

Info from Step 1 in blue

Info from Step 2 in red

Info from Step 3 in green

graph of $f(x) = \dfrac{1}{x^2+25}$



end of Example

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end of Lecture