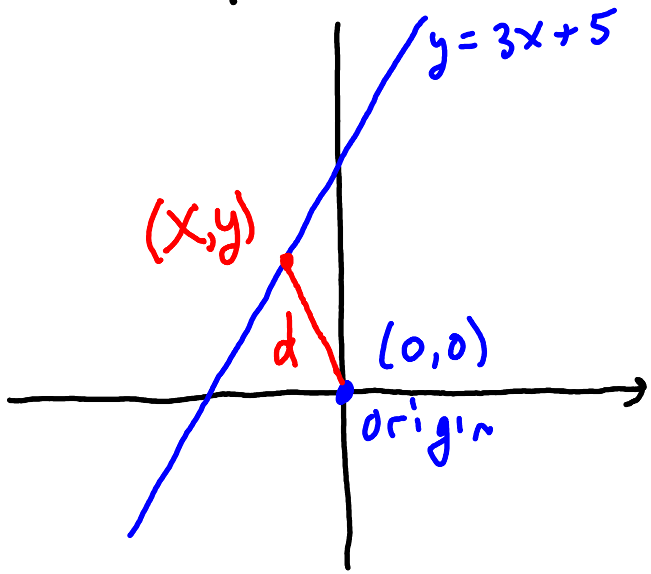
Section 4.5 Optimization Problems

Optimization Problems are simply absolute max/min problems, but with certain complications.

- Often Presented as a word problem
- variables, equations are not given to you you have to figure them out.
- Often more than one variable. In order to use our max/min techniques, you'll have to figure out how to eliminate all but one variable

[Example 1] (Similar to 4.1 #15)

Find the point on the line $y=3x+5$ that is closest to the origin.



(Find point (x,y) on the line that minimizes d.)

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(x-0)^2+(y-0)^2}$$

$$d=\sqrt{x^2+y^2}$$

We know $y=3x+5$ because point (x,y) must be on the line

Substitute that in

$$d=\sqrt{x^2+(3x+5)^2}=\sqrt{x^2+9x^2+30x+25}=\sqrt{10x^2+30x+25}$$

Find the value of x that minimizes d. Then use that x to get y.

Strategy: Find critical numbers for d.

We expect to find one. It will be the min.

$$d'(x) = \frac{d}{dx}\sqrt{10x^2+30x+25}$$

$$= \frac{1}{2\sqrt{10x^2+30x+25}} \cdot (20x+30)$$

$$d'(x) = \frac{10x+15}{\sqrt{10x^2+30x+25}}$$

denominator is d

Chain Rule details

$\text{inner}(x) = 10x^2+30x+25$

$\text{inner}'(x) = 20x+30$

$\text{outer}(\ ) = \sqrt{\ } = (\ )^{1/2}$

$\text{outer}'(\ ) = \frac{1}{2}(\ )^{\frac{1}{2}-1} = \frac{1}{2}(\ )^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{(\ )^{1/2}}$

$= \frac{1}{2\sqrt{(\ )}}$

Are there any x values that cause $d'(x)$ to not exist?

Notice the denominator of $d'(x)$ is d, which will never be 0 because line does not go through origin.

Any x values to cause $d'(x)=0$?

Set $0 = \text{numerator} = 10x+15$

Result $x = -3/2$ will cause numerator $= 0$

Since denominator $\neq 0$, we know $d'(-\frac{3}{2}) = 0$

And we know $d(-\frac{3}{2})$ exists because $d(x)$ is always positive

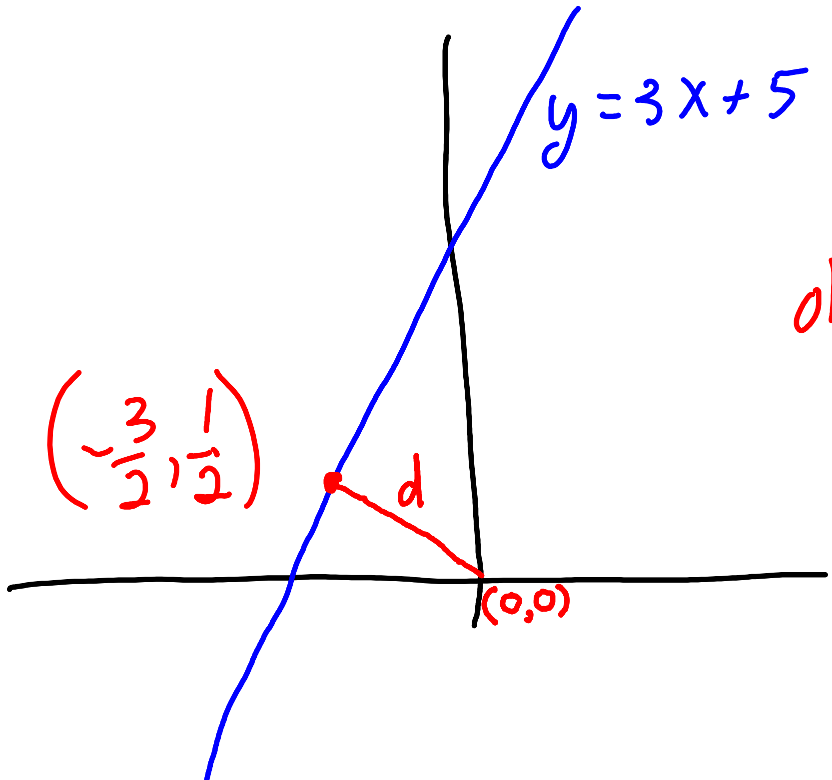
So we can see that $x = -3/2$ is the only critical number for $d(x)$

We discussed how we know this critical number $x = -3/2$ will <u>minimize</u> d.

The corresponding value of y is

$$y = 3x + 5 = 3\left(-\frac{3}{2}\right) + 5 = -\frac{9}{2} + 5 = -\frac{9}{2} + \frac{10}{2} = \frac{1}{2}$$

So the point $(x, y) = \left(-\frac{3}{2}, \frac{1}{2}\right)$ is closest to the origin.



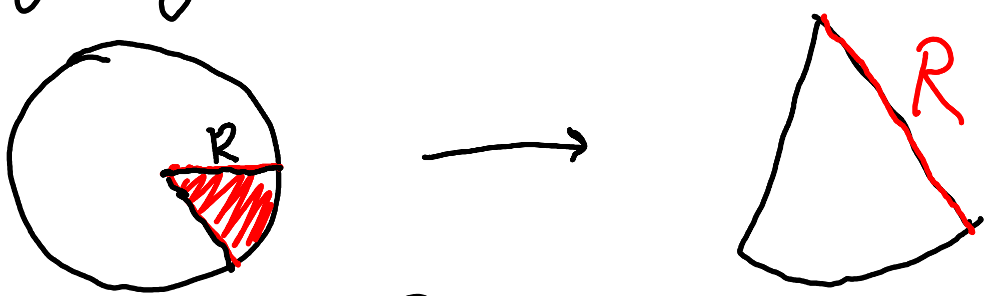
observe: Slope of red segment is

$$m = \frac{\Delta y}{\Delta x} = \cdots = -\frac{1}{3}$$

So red segment is <u>perpendicular</u> to blue line.

end of example

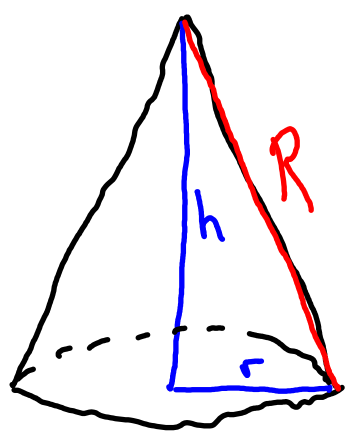
[Example 2] A conical drinking cup is to be made by cutting a sector out of a circular disk of paper, and then gluing the remaining paper into a cone



Disk has radius R.

What is largest possible volume of resulting cone?

<u>Solution</u>



$$V_{cone} = \frac{1}{3}\pi r^2 h$$

$$r^2 + h^2 = R^2$$

Use second equation to eliminate r^2

$$r^2 = R^2 - h^2$$

Substitute into Volume equation

$$V_{cone} = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(R^2 - h^2)h$$

$$= \frac{\pi}{3}\left(R^2 h - h^3\right)$$

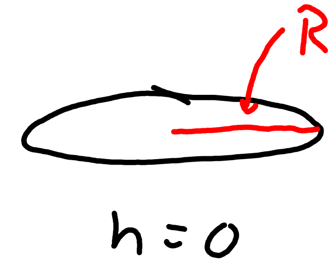
Find value of h that maximizes V_{cone}

Remark:

We expect that there will be one value of h that maximizes V. The reasoning is

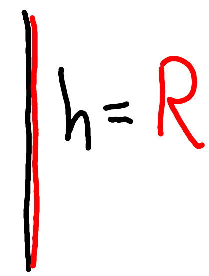
One extreme:

If $h=0$, then the cone is just a flat disk, which would have no volume

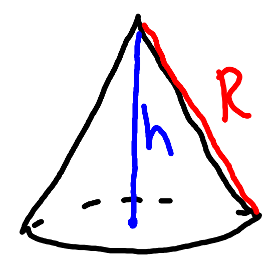


The other extreme:

If $h=R$, then the cone is just a vertical line, which would have no volume



In between, there should be one optimal value of h.



Strategy: Find $V'(h)$

Set $V'(h) = 0$,
Solve for h

$\frac{1}{3}\pi(R^2 h - h^3)$

Find V'

$$V = \frac{\pi}{3}(R^2 h - h^3)$$

$$V' = \frac{d}{dh}\frac{\pi}{3}(R^2 h - h^3) = \frac{\pi}{3}\frac{d}{dh}(R^2 h - h^3)$$

$$= \frac{\pi}{3}(R^2 - 3h^2)$$

Set $V'=0$ and solve for h

$$0 = V' = \frac{\pi}{3}\left(R^2 - 3h^2\right)$$

$$0 = R^2 - 3h^2$$

$$3h^2 = R^2$$

$$h^2 = \frac{R^2}{3}$$

$$h = \pm \frac{R}{\sqrt{3}}$$

We know that the height h must be positive

Conclude that the optimal h is $h = \frac{R}{\sqrt{3}}$

The resulting Cone Volume is

$$V_{max} = \frac{\pi}{3}\left(R^2 h - h^3\right)$$

$$= \frac{\pi}{3}\left(R^2\left(\frac{R}{\sqrt{3}}\right) - \left(\frac{R}{\sqrt{3}}\right)^3\right)$$

$$= \frac{\pi}{3}\left(\frac{R^3}{\sqrt{3}} - \frac{R^3}{3\sqrt{3}}\right)$$

$$= \frac{\pi}{3}\left(\frac{2R^3}{3\sqrt{3}}\right)$$

$$= \frac{2\pi R^3}{9\sqrt{3}}$$

end of Example

End of lecture