MATH 2301 (Barsamian) Lecture #27 (Mon Nov 6, 2023)

Pick Up Graded Work

Sign In

Sit next to somebody: You will be working in pairs.

Today: Section 4.6 Newton's Method

Tues Recitation: Section 4.5 Optimization
Section 4.6 Newton's Method

Wednesday: Section 4.7 Antiderivatives

Friday: No Class

Next Monday (Nov 13): Exam X3 Covering Chapter 4

Newton's Method (Section 4.6)

Recall that a root of a function $f(x)$ is an x value $x = r$ such $f(r) = 0$. So that the point $(r, f(r))$ is an x intercept on the graph of $f(x)$.

Newton's Method is a way of finding an approximate value for a root r. The method starts with an initial guess, called x_1.

If x_1 is a root (that is if $f(x_1) = 0$) then you're done.

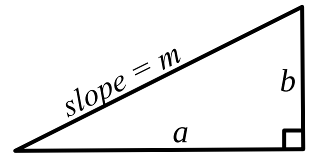
But if x_1 is not a root, then Newton's Method comes up with a next guess, x_2, that should be a better guess. That is $f(x_2)$ should be even closer to 0 than $f(x_1)$.

And so on. You obtain a list $\underset{\uparrow}{x_1}, \underbrace{x_2, x_3, \ldots}_{\text{from Newton's Method}}$. An iterative process.

your initial guess

Class Drill Part 1: The Idea Behind Newton's Method

(a) In the triangle shown, find an equation for the slope m of the hypotenuse in terms of the lengths a and b.



$m = \frac{b}{a} = \frac{\text{rise}}{\text{run}}$

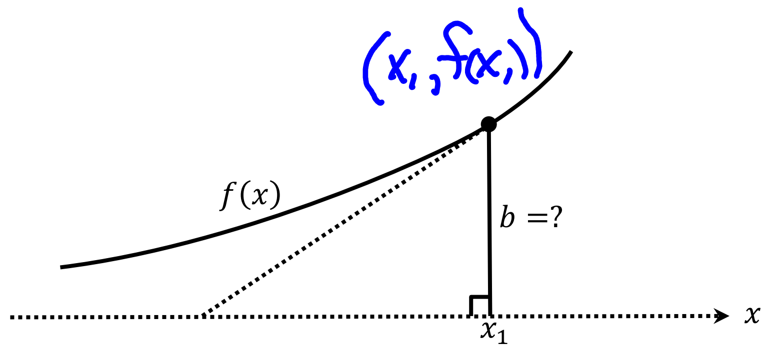
(b) Solve the equation for a in terms of m and b:

$a = \frac{b}{m}$

(c) In the triangle shown, the upper right vertex lies on the graph of f.

How tall is the right leg?

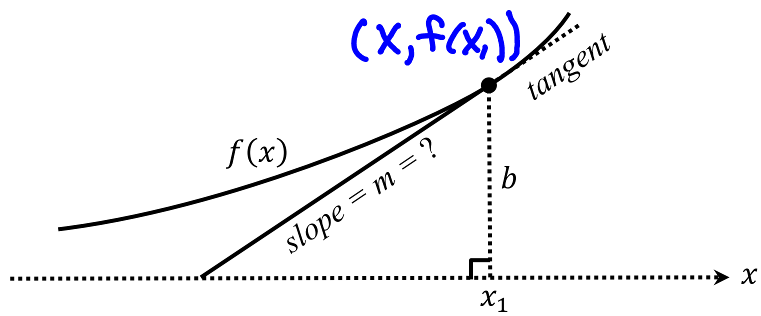
$b = f(x_1)$



(d) Suppose that the hypotenuse of the triangle is known to lie on the line that's tangent to the graph of f at the point where $x = x_1$

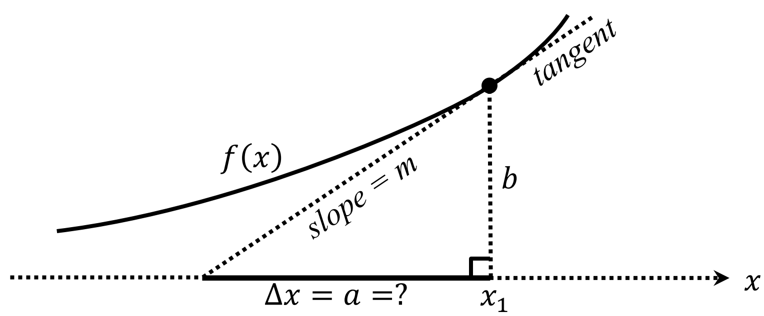
What is the hypotenuse slope m?

$m = f'(x_1)$



(e) For the same triangle, what is the base Δx?

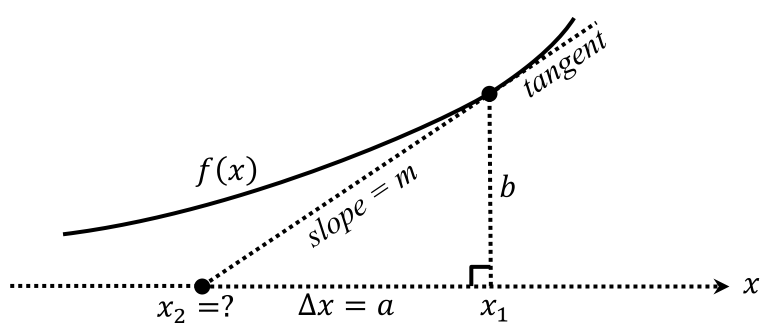
$\Delta x = a = \frac{b}{m} = \frac{f(x_1)}{f'(x_1)}$



(f) For the same triangle, what is the x coordinate x_2?

$x_2 = x_1 - \Delta x$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$



The Class Drill continues on the next page ➔

Class Drill Part 2: Using Newton's Method

Newton's Method

Given: A function f that is differentiable on an interval I and that has a root in I. That is, it is known that there exists a number r somewhere in I such that $f(r) = 0$.

Goal: Find an approximate value for the root r, accurate to d decimal places.

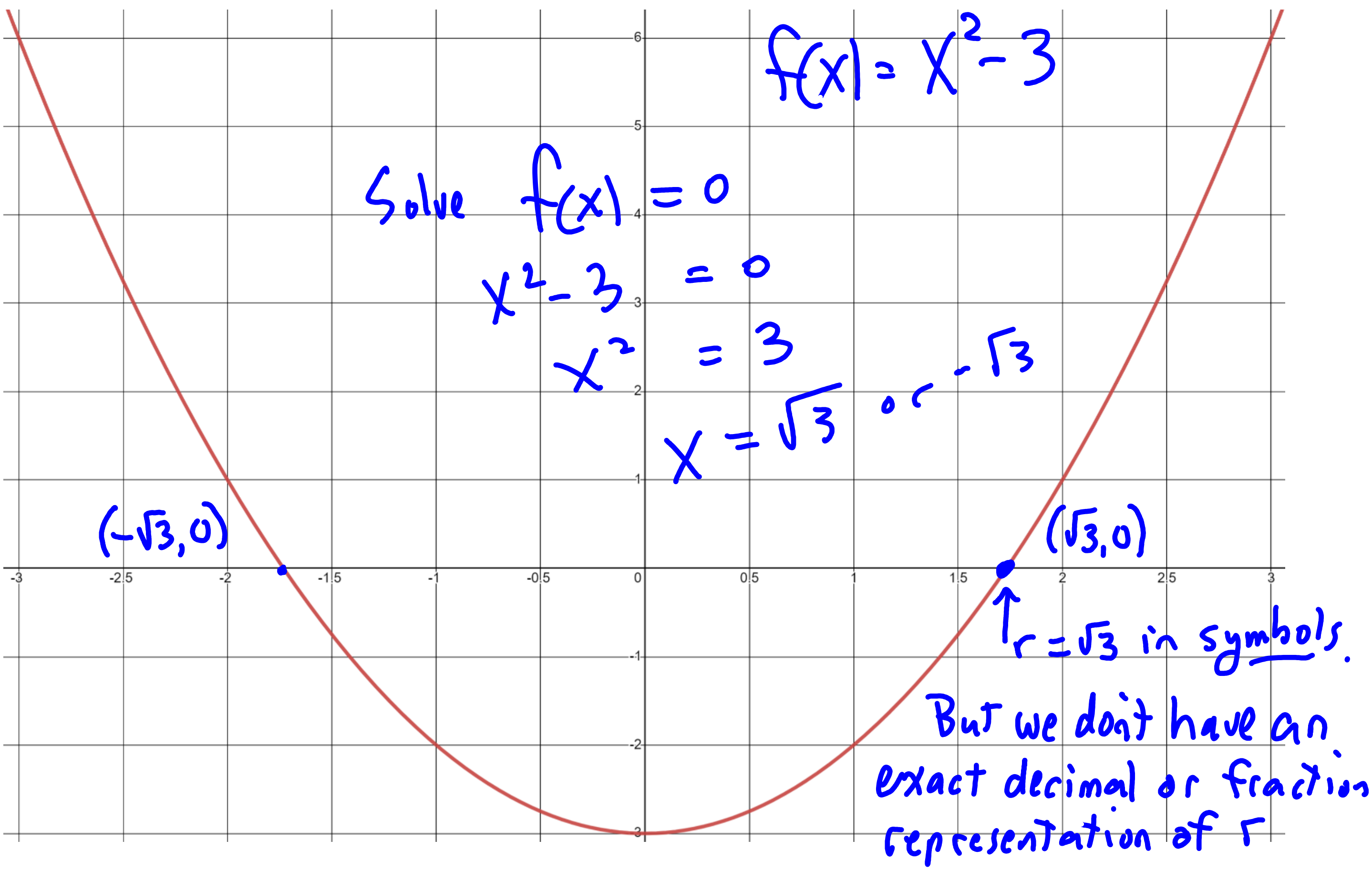
Step 1: Choose a value x_1 as an initial approximation of the root. (This is often done by looking at a graph.)

Step 2: Create successive approximations iteratively, as follows:
Given an approximation x_n, compute the next approximation x_{n+1} by using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Step 3: Stop the iterations when successive approximations do not differ in the first d places after the decimal point. The last x value computed is the approximation of r.

Let $f(x) = x^2 - 3$. Observe that the graph of $f(x)$ shows an x intercept somewhere between $x = 1$ and $x = 2$. Using the terminology of roots, we would say that there is a root of f, that is, a number r such that $f(r) = 0$, and that r is somewhere between 1 and 2.



The goal is to use Newton's method to find an approximation for the root r. You will do the first three iterations only, using the initial approximation $x_1 = 3$. That is, you will find x_2, x_3, x_4.

For the function $f(x) = x^2 - 3$,

(5)

(a) Compute $f'(x)$ $f'(x) = 2x$

(b) Fill out the following table. (Do the details below.)

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	$x_1 = 3$	6	6	$x_2 = 3 - \frac{6}{6} = 3 - 1 = 2$
2	$x_2 = 2$	1	4	$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2 - \frac{1}{4} = \frac{7}{4} = 1.75$
3	$x_3 = \frac{7}{4}$	$\frac{1}{16}$	$\frac{7}{2}$	$x_4 = \frac{7}{4} - \frac{1/16}{7/2} = \frac{7}{4} - \frac{1}{56} =$
4	$x_4 = \frac{97}{56}$	≈ 1.73214		

$\frac{7(14)}{4(14)} - \frac{1}{56} = \frac{98}{56} - \frac{1}{56}$

$f(x_1) = f(3) = 3^2 - 3 = 6$

$f'(x_1) = f'(3) = 2(3) = 6$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{6}{6} = 3 - 1 = 2$

$f(x_2) = f(2) = 2^2 - 3 = 4 - 3 = 1$

$f'(x_2) = f'(2) = 2(2) = 4$

$f(x_3) = f(\frac{7}{4}) = (\frac{7}{4})^2 - 3 = \frac{49}{16} - 3 = \frac{49}{16} - \frac{48}{16} = \frac{1}{16}$

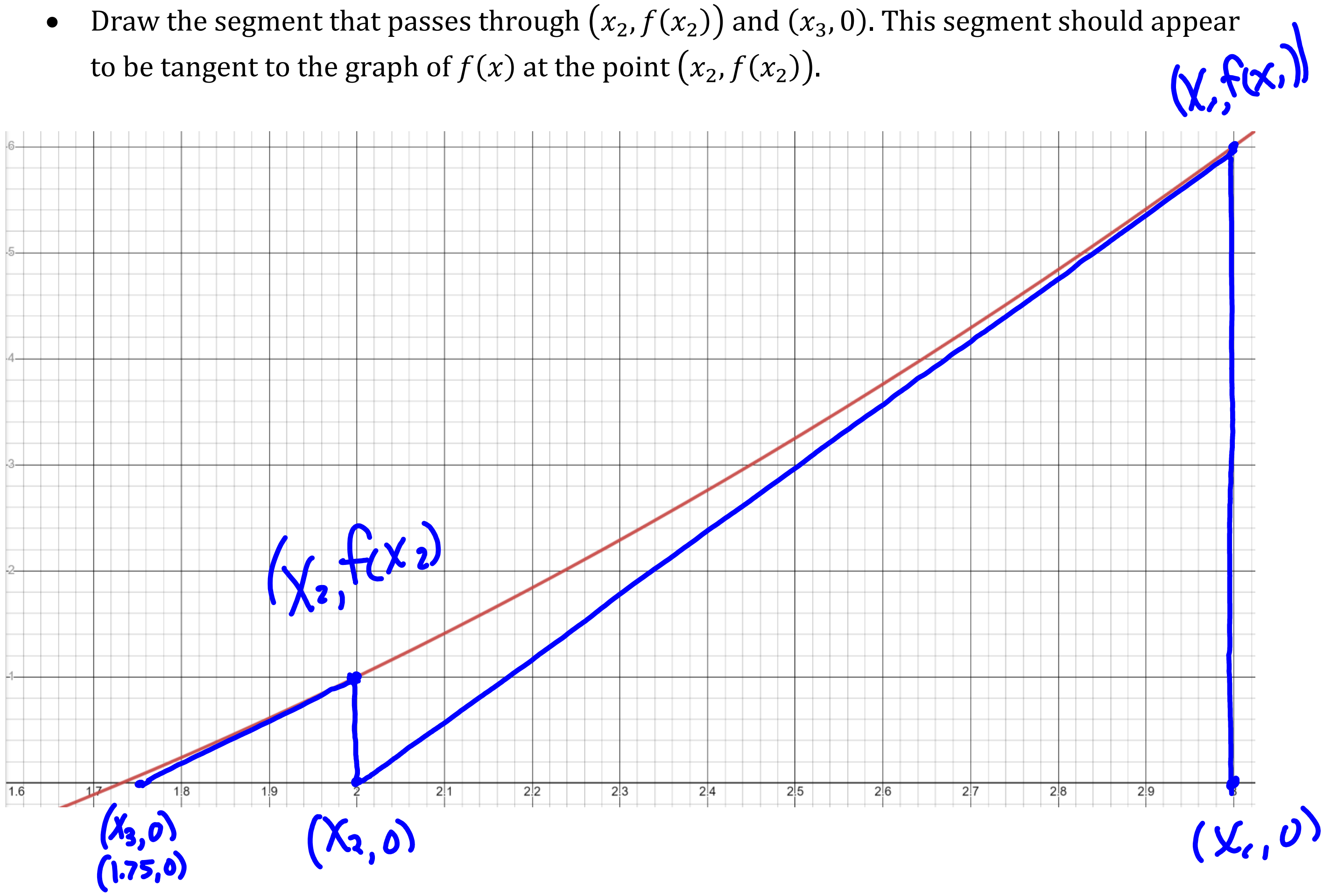
$f'(x_3) = f'(\frac{7}{4}) = 2(\frac{7}{4}) = \frac{7}{2}$

The Class Drill continues on the next page ➔

$\frac{1/16}{7/2} = \frac{1}{\cancel{16}} \cdot \frac{\cancel{2}}{7} = \frac{1}{8 \cdot 7} = \frac{1}{56}$

(C) A zoomed-in graph of $f(x)$ is shown below. You'll illustrate some of your results on this graph.

- Put a point at $(x_1, 0)$
- Put a point at $(x_1, f(x_1))$
- Draw the segment that connects $(x_1, 0)$ and $(x_1, f(x_1))$. This segment should be vertical.
- Put a point at $(x_2, 0)$.
- Draw the segment that passes through $(x_1, f(x_1))$ and $(x_2, 0)$. This segment should appear to be tangent to the graph of $f(x)$ at the point $(x_1, f(x_1))$.
- Put a point at $(x_2, f(x_2))$
- Draw the segment that connects $(x_2, 0)$ and $(x_2, f(x_2))$. This segment should be vertical.
- Put a point at $(x_3, 0)$.
- Draw the segment that passes through $(x_2, f(x_2))$ and $(x_3, 0)$. This segment should appear to be tangent to the graph of $f(x)$ at the point $(x_2, f(x_2))$.



Check: find $f(x_4)$

$$f(x_4) = f\left(\frac{97}{56}\right) = \left(\frac{97}{56}\right)^2 - 3 = \frac{9409}{3136} - \frac{3(3136)}{(3136)} = \frac{9409}{3136} - \frac{9408}{3136}$$

$$= \frac{1}{3136}$$

$$\begin{array}{r} 97 \\ 97 \\ \hline 679 \\ 8730 \\ \hline 9409 \end{array} \qquad \begin{array}{r} 56 \\ 56 \\ \hline 336 \\ 2800 \\ \hline 3136 \end{array} \qquad \begin{array}{r} 3136 \\ 3 \\ \hline 9408 \end{array}$$

Very close to 0!!

Our list of X values

$$\underset{\text{initial guess}}{X_1 = 3} \quad \underbrace{X_2 = 2, \quad X_3 = \frac{7}{4} = 1.75, \quad X_4 = \frac{97}{56}, \ \ldots}_{\text{from Newton's method}}$$

These numbers are getting closer and closer to the root $r = \sqrt{3}$

$\boxed{\text{End of Lecture}}$