MATH 2301 (Barsamian) Lecture #28 (Wed Nov 9, 2023) ①

Pick Up Graded Papers

Sign In

Today: Section 4.7 Antiderivatives

Today 4pm: Review Session in Morton 223

Thurs 4pm: Review Session in Morton 223

Sun 4pm: Review Session in Morton 223

Friday: No class

Monday: Exam X3, Covering Chapter 4

In today's meetings of Section 100 and Section 110, I covered slightly different material. Students from both sections should read all of these notes to get all of the material.

Section	L A S T N A M E	,	F I R S T N A M E

Quiz Q7, Fri Nov 3, 2023 (20 min)
Fall 2023 MATH 2301 (Barsamian)
No books, notes, calculators, phones.

Problem:	[1]	[2]	[3]	Total	%
Your Score:					
Possible:	10	10	10	30	100%

[1] (10 points)

The Mean Value Theorem

If a function f and an interval $[a,b]$ satisfy the following two requirements (the ***hypotheses***)

- f is *continuous* on the *closed interval* $[a,b]$
- f is *differentiable* on the *open interval* (a,b)

then the following statement (the ***conclusion***) is true:

There is a number $x = c$ (at least one) with $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

In other words,

The slope of the *tangent line* at *c* equals the slope of the *secant line* from *a* to *b*.

Remark: The theorem does not give you the value of c. If a c exists, you have to figure out its value.

The function $f(x) = \dfrac{1}{x}$ and the interval $[1,4]$ do satisfy the two *hypotheses* of the Mean Value Theorem. Given

(a) Find all numbers c that satisfy the *conclusion* of the Mean Value Theorem. Show all details clearly.

Set up the equation and solve it for c.

Left side: we need $f'(c)$

Start by getting $f'(x)$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = (-1)x^{-1-1} = -1x^{-2} = \frac{-1}{x^2}$

$f'(c) = -\frac{1}{c^2}$

Right side $\frac{f(4)-f(1)}{4-1} = \frac{\frac{1}{4}-1}{3} = \frac{\frac{-3}{4}}{3} = -\frac{\cancel{3}}{4}\cdot\frac{1}{\cancel{3}} = \frac{-1}{4}$

equate left side = right side

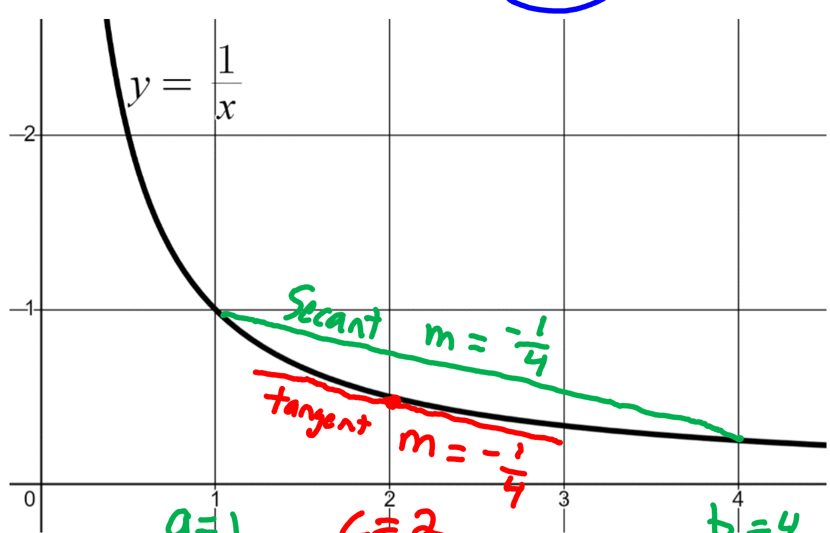
$\frac{-1}{c^2} = -\frac{1}{4}$

$c^2 = 4$

$c = -2$ or $\boxed{c = 2}$ must have $1 < c < 4$

(b) Illustrate your result on the given graph of $f(x)$.

Remark: It is possible to make a reasonable illustration for **(b)** even if you are unable to do **(a)**.



The Quiz continues on back ➔

[2](10 points) (Suggested Exercise 4.3#1) Let $f(x) = x^3 - 12x + 5$

(a) Using Calculus, find the intervals on which f is *increasing* and the intervals on which f is *decreasing*. Show all details clearly. (You must use Calculus. No credit for just finding y values at a bunch of x values.)

Study sign behavior of $f'(x) = 3x^2 - 12 = 3(x^2-4) = 3(x+2)(x-2)$

Sign chart for $f'(x)$

$+ + + +$ $f'=0$ $- - -$ $f'=0$ $+ + +$

$x=-3$ | -2 max | $x=0$ | 2 min | $x=3$

critical $x=-2$, $x=2$

$f'(-3) = 3((-3)+2)((-3)-2) = (+)(-)(-) = \text{pos}$

$f'(0) = 3((0)+2)((0)-2) = (+)(+)(-) = \text{neg}$

$f'(3) = 3((3)+2)((3)-2) = (+)(+)(+) = \text{pos}$

f increasing on $(-\infty, -2]$ and $[2, \infty)$

f decreasing on $[-2, 2]$

(b) Find all *local maximum values* and *local minimum values* of f. Show all details clearly.

$y_{max} = f(-2) = (-2)^3 - 12(-2) + 5 = \cdots = 21$

$y_{min} = f(2) = (2)^3 - 12(2) + 5 = \cdots = -11$

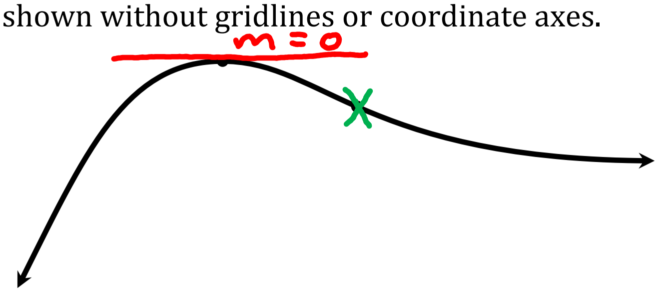
[3] (10 points) The graph of a function $f(x)$ is shown without gridlines or coordinate axes.

The formulas for $f(x)$ and its derivatives are

$$f(x) = xe^{(-x)}$$

$$f'(x) = -(x-1)e^{(-x)}$$

$$f''(x) = (x-2)e^{(-x)}$$



(A) There is a highest point. What are its (x, y) coordinates? Explain clearly how you know.
(You don't have to *prove* that there is a high point. You just need to explain how you know where it is.)

high point happens where $f'(x) = 0$. So set $f'(x) = 0$, Solve for x.

$0 = f'(x) = -\underbrace{(x-1)}_{\text{this must be } 0}\underbrace{e^{(-x)}}_{\text{never zero}}$

So $x = 1$

So $y = f(1) = 1e^{-1} = 1 \cdot \frac{1}{e} = \frac{1}{e}$

(B) There is an inflection point. What are its (x, y) coordinates? Explain clearly how you know.
(You don't have to *prove* that there is an inflection point. Just explain how you know where it is.)

inflection point happens where $f''(x) = 0$

$0 = f''(x) = \underbrace{(x-2)}_{\text{this must be zero}}\underbrace{e^{(-x)}}_{\text{never zero}}$

So $x = 2$

Then $y = f(2) = 2e^{(-2)} = 2 \cdot \frac{1}{e^2} = \frac{2}{e^2}$

Antiderivatives (Section 4.7)

Definition of Antiderivative

words: $F(x)$ is an antiderivative of $f(x)$

capital ... lower case

meaning: $f(x)$ is the derivative of $F(x)$

lower case ... capital

$$F'(x) = f(x)$$

diagram

$$F(x) \xrightarrow{\text{take derivative}} f(x)$$

Example

(a) Is $F(x) = \dfrac{x^3}{3}$ an antiderivative of $f(x) = x^2$?

Solution Check

$$F'(x) = \frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{1}{3}\frac{d}{dx}x^3 = \frac{1}{3}\cdot 3x^2 = x^2 = f(x).$$

yes!

(b) Is $G(x) = \dfrac{(x^2+5x+3)^3}{3}$ an antiderivative of $g(x) = (x^2+5x+3)^2$?

Check:

$$G'(x) = \frac{d}{dx}\frac{(x^2+5x+3)^3}{3} = \frac{1}{3}\frac{d}{dx}\underbrace{(x^2+5x+3)^3}_{\text{chain rule}} = \frac{1}{\cancel{3}}\left[\cancel{3}(x^2+5x+3)^2\cdot(2x+5)\right]$$

$$= (x^2+5x+3)^2\cdot(2x+5) \neq g(x)$$

So $G(x)$ is <u>not</u> an antiderivative of $g(x)$

(c) Is $H(x) = \dfrac{x^3}{3} + 17$ an antiderivative of $f(x) = x^2$?

Check $H'(x) = \dfrac{d}{dx}\left(\dfrac{x^3}{3} + 17\right) = x^2 + 0 = x^2 = f(x)$ ✓ yes

Particular and General Antiderivatives

If $F(x)$ is an antiderivative of $f(x)$ then any other function of the form $F(x) + C$ will also be an antiderivative.

real number constant

For the function $f(x) = x^2$

$F(x) = \frac{x^3}{3}$ is a Particular antiderivative

$H(x) = \frac{x^3}{3} + 17$ is also a Particular antiderivative

these are actual functions

The General Antiderivative of $f(x) = x^2$ is the function form $\frac{x^3}{3} + C$ ← C represents a real number constant.

The General Antiderivative is called a function form, not a function, because C has not been chosen. Once a value is chosen for C, the expression becomes an actual function that is a Particular Antiderivative.

Derivative Rules

Rule	function	derivative
Power Rule	x^n	nx^{n-1}
Sum and constant multiple	$af(x)+bg(x)$	$af'(x)+bg'(x)$
trig	$\sin(x)$	$\cos(x)$
	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\sec^2(x)$
exponentials and logs	$e^{(x)}$	$e^{(x)}$
	$\ln(x)$	$\frac{1}{x}$
	$x\ln(x)-x$	$\ln(x)$

Antiderivative rules

Rule	function	particular antiderivative
Power Rule	x^n with $X \neq -1$	$\frac{x^{n+1}}{n+1}$
Sum + constant multiple	$af(x)+bg(x)$	$aF(x)+bG(x)$
trig	$\cos(x)$	$\sin(x)$
	$\sin(x)$	$-\cos(x)$
	$\sec^2(x)$	$\tan(x)$
exponentials	$e^{(x)}$	$e^{(x)}$
$1/x$	$\frac{1}{x}$	$\ln(\lvert x\rvert)$
logarithms	$\ln(x)$	$x\ln(x)-x$

Examples

(a) $f(x) = x^2$ $\leftarrow n=2$ Find an antideriv using the rules

Solution $F(x) = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$

(b) $f(x) = \frac{1}{x^2}$ Find the general antiderivative using the rules

Solution rewrite $f(x) = \frac{1}{x^2} = x^{-2}$

General antideriv $F(x) = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C = (-1)\cdot\frac{1}{x} + C$

$= -\frac{1}{x} + C$

(c) $g(x) = x$ $\leftarrow x = x^1 \quad n=1$ find general antideriv

Solution $G(x) = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$

d) $h(x) = 1$ $\quad 1 = x^0 \quad n = 0$ find general antideriv

$$H(x) = \frac{x^{0+1}}{0+1} + C = \frac{x}{1} + C = x + C$$

e) $f(x) = \frac{1}{x}$

try power rule $\cancel{f(x) = \frac{1}{x} = x^{-1}}$ $\quad \cancel{F(x) = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C}$

$\cancel{= \frac{1}{0} + C}$ ~~undefined~~

power rule does not apply when $x = -1$

The general antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln(|x|) + C$

using the Antiderivative rule for $1/x$ from the table.

(f) $f(x) = 5x^{3} - 4x^{7}$ $\leftarrow n=3$ $\leftarrow n=7$

(a) Find the General Antiderivative

Solution Use the Sum and Constant Multiple Rule and the Power Rule

$$F(x) = 5\cdot\frac{x^{3+1}}{3+1} - 4\cdot\frac{x^{7+1}}{7+1} + C = \frac{5\cdot x^{4}}{4} - \frac{4x^{8}}{8} + C = \boxed{\frac{5x^{4}}{4} - \frac{x^{8}}{2} + C = F(x)}$$

General Antiderivative

(b) Find the Particular Antiderivative $F(x)$ such that $F(1)=10$.

Solution: Turn the equation around

$$10 = F(1) = \frac{5(1)^{4}}{4} - \frac{(1)^{8}}{2} + C = \frac{5}{4} - \frac{1}{2} + C = \frac{3}{4} + C$$

Subtract $\frac{3}{4}$ from both sides

$$\frac{37}{4} = C$$

Now use this value of C to build the Particular Antiderivative

$$\boxed{F(x) = \frac{5x^{4}}{4} - \frac{x^{8}}{2} + \frac{37}{4}}$$ Particular Antiderivative

end of lecture