MATH 2301 (Barsamian) Lecture #35 (Mon Dec 4, 2023)

Pick up Handout on Substitution

Sign In

Today: Section 5.5 Substitution Method

Tomorrow Recitation: Sections 5.4, 5.5

Wednesday: Section 5.5

Friday: Review

Thurs Dec 14 2:30-4:30pm Final Exam

More details later

Section 5.5 The Substitution Method

Used for finding $\int f(x)\,dx$ in certain cases where the integrand $f(x)$ involves a nested function.

The Method of Integration by Substitution

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Remember that the *Chain Rule for Derivatives* is used for taking the *derivative* of *nested functions*:

Chain Rule for Derivatives: $\frac{d}{dx} outer\big(inner(x)\big) = outer'\big(inner(x)\big) \cdot inner'(x)$

The goal now is to find the *general antiderivative* of a function $f(x)$ that involves a *nested function*. That is, we wish to find the *indefinite integral* $\int f(x)\,dx$ where the integrand $f(x)$ involves a nested function. This is not always possible. But sometimes it is, using the *Substitution Method*.

The Substitution Method for finding the *indefinite integral* $F(x) = \int f(x)\,dx$ where the integrand $f(x)$ involves a *nested function*.

Step 1 Identify the inner function and call it u. Write the equation $inner(x) = u$ to introduce the single letter u to represent the inner function. Circle the equation.

Step 2 Build the equation $dx = \frac{1}{u'}du$. To do this, first find u', then use it to build equation $dx = \frac{1}{u'}du$. Circle the equation.

Step 3 Substitute, Cancel, Simplify. In steps (1) and (2) you have two circled equations. **Substitute** these into the integrand of your indefinite integral. **Cancel** as much as possible and **simplify** by using the *Constant Multiple Rule*. The result should be a new basic integral involving just the variable u. (See **Remarks about Step 3** below.)

Step 4 Integrate. Find the new indefinite integral by using the indefinite integral rules. The result should be a *function form* involving just the variable u and $+C$.

Step 5 Substitute Back. Substitute $u = inner(x)$ into your function from Step (4) The result will be a new function form involving just the variable x and the $+C$. This is the $F(x)$ that we seek. Present the result clearly as $F(x) = BLAH$ and circle it.

Remarks about Step 3: The result of **Step 3** should be a new indefinite integral with an integrand that is a function involving the variable u. There are three important things to check at the end of **Step 3**:

- There should be no x in the new indefinite integral. It should involve only u.
- The new indefinite integral should *not* involve a *nested function*, and it should be a *basic integral* that can be integrated using our indefinite integral rules.
- If the above two items are not satisfied, then either you made a mistake, or the original integral might be one for which the Substitution Method cannot be used.

[Example 1] Find $\int x \cos x^2 dx$

Solution Use Substitution Method

Step 1 Identify Inner Function

More helpful typesetting $\int x \cos(x^2) dx$

$$x^2 = u$$

Step 2 Build equation $dx = \frac{1}{u'} du$

need $u' = \frac{d}{dx} u = \frac{d}{dx} x^2 = 2x$

Now build $dx = \frac{1}{2x} du$

Step 3 Substitute, Cancel, Simplify

$$\int x \cos(x^2)\, dx \underset{\substack{\uparrow \\ \text{substitute}}}{=} \int x \cos(u) \frac{1}{2x} du \underset{\substack{\uparrow \\ \text{cancel}}}{=} \int \cos(u) \cdot \frac{1}{2} \cdot du \underset{\substack{\uparrow \\ \text{simplify}}}{=} \frac{1}{2} \int \cos(u)\, du$$

Step 4 Integrate

$$\frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C$$

Step 5 Substitute Back

$$\boxed{\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C}$$

End of [Example 1]

[Example 2] Find $\int \frac{(\ln(x))^5}{x}\,dx$

Solution

Step 1 Identify Inner Function

$\ln(x) = u$

Step 2 Build $dx = \frac{1}{u'}du$

we need $u' = \frac{du}{dx} = \frac{d}{dx}\ln(x) = \frac{1}{x}$

Build equation $dx = \frac{1}{1/x}du$

$dx = x\,du$

Step 3 Substitute, Cancel, Simplify

$$\int \frac{(\ln(x))^5}{x}\,dx = \int \frac{(u)^5}{x}\,x\,du = \int u^5\,du$$

substitute

cancel

Step 4 Integrate

$n=5$

$$\int u^5\,du = \frac{u^{5+1}}{5+1} + C = \frac{u^6}{6} + C$$

Step 5 Sub Back

$$\boxed{\int \frac{(\ln(x))^5}{x}\,dx = \frac{(\ln(x))^6}{6} + C}$$

[Example 3] Find $\int \frac{\sec^2(\pi/x)}{x^2}\,dx$

Step 1 Identify Inner Function

$$\frac{\pi}{x} = u$$

$n=-1$

Step 2 Build $dx = \frac{1}{u'}du$

need $u'(x) = \frac{d}{dx}u = \frac{d}{dx}\left(\frac{\pi}{x}\right) = \frac{d}{dx}(\pi x^{-1}) = \pi \frac{d}{dx} x^{-1} = \pi(-1)x^{-1-1}$

$$= -\pi x^{-2} = -\frac{\pi}{x^2}$$

Now build equation $dx = \frac{1}{-\pi/x^2}\,du$

$$dx = -\frac{x^2}{\pi}\,du$$

Step 3 Sub, Cancel, Simplify

$$\int \frac{\sec^2(\pi/x)}{x^2}\,dx \underset{\text{sub}}{=} \int \frac{\sec^2(u)}{x^2}\left(-\frac{x^2}{\pi}\,du\right)$$

$$\underset{\text{cancel}}{=} \int \sec^2(u)\cdot -\frac{1}{\pi}\,du$$

$$\underset{\text{simplify}}{=} -\frac{1}{\pi}\int \sec^2(u)\,du$$

Step 4 Integrate

$$-\frac{1}{\pi}\int \sec^2(u)\,du = -\frac{1}{\pi}\tan(u) + C$$

Step 5 Substitute Back

$$\boxed{\int \frac{\sec^2(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\tan\left(\frac{\pi}{x}\right) + C}$$

Recall Derivative

$$\frac{d}{dx} e^{(kx)} = e^{(kx)} \cdot k$$

Chain Rule Details

$\text{inner}(x) = kx$

$\text{inner}'(x) = k$

$\text{outer}(\;) = e^{(\;)}$

$\text{outer}'(\;) = e^{(\;)}$

$$\boxed{\frac{d}{dx} e^{kx} = ke^{kx}}$$

[Example 4] Find $\int e^{kx}\,dx$

<u>Solution</u> Integral would be clearer as $\int e^{(kx)}\,dx$

$$\int e^{(kx)}\,dx \underset{\text{substitute}}{=} \int e^{(u)}\,\frac{1}{k}\,du \underset{\text{Simplify}}{=} \frac{1}{k}\int e^{u}\,du \underset{\text{integrate}}{=} \frac{1}{k}e^{u}+C \underset{\text{Sub back}}{=} \frac{1}{k}e^{(kx)}+C$$

$kx = u$

$dx = \frac{1}{k}\,du$

because

$u = kx$

so $u' = \frac{d}{dx}(kx) = k$

$dx = \frac{1}{u'}\,du$

$dx = \frac{1}{k}\,du$

Conclusion

$$\boxed{\int e^{(kx)}\,dx = \frac{1}{k}e^{(kx)}+C}$$

[Example 5] $\int x e^{(-x^2)} dx$

Solution

Step 1 $-x^2 = u$

Step 2 Build $dx = \frac{1}{u'} du$

need $u' = \frac{d}{dx} u = \frac{d}{dx} -x^2 = -2x$

$dx = \frac{1}{-2x} du$

Step 3 Sub, Cancel, Simplify

$$\int x e^{(-x^2)} dx \underset{\text{Sub}}{=} \int x e^{u} \left(\frac{1}{-2x} du \right) \underset{\text{cancel}}{=} \int e^{u} \cdot \frac{1}{-2} du \underset{\text{Simplify}}{=} -\frac{1}{2} \int e^{u} du$$

Step 4 Integrate

$$-\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} + C$$

Step 5 Substitute back

$$\int x e^{(-x^2)}\,dx = -\frac{1}{2}e^{(-x^2)} + C$$

[Example 6] Find $\int e^{(-x^2)}dx$

Solution

Integrand is nested. Try Substitution

Step 1 $\quad -x^2 = u$

Step 2 $\quad dx = -\frac{1}{2x}du$ Same as previous example

Step 3 Sub, cancel simplify

$$\int e^{(-x^2)}dx \underset{\text{Sub}}{=} \int e^u\left(\frac{-1}{2x}du\right) \underset{\text{Simplify}}{=} -\frac{1}{2}\int \frac{e^{(u)}}{x}du$$

Integral involves both x + u. Cannot proceed

Substitution method Fails!

Question: How can we find an antiderivative of $e^{(-x^2)}$

Extremely Important Function!

Bell Shaped Curve

Good News there is an antiderivative!

Bad News Antideriv cannot be written as a combination of basic functions.

Good News We can build an antiderivative using an area function

$$f(x) = e^{(-x^2)} \qquad F(x) = \int_{t=0}^{t=x} e^{-t^2}\,dt$$

So we could write the general antiderivative as

$$\int e^{(-x^2)}\,dx = \int_{t=0}^{t=x} e^{-t^2}\,dt + C$$

So substitution method fails, but there is another way to write an antiderivative

$\boxed{\text{End of Lecture}}$