MATH 2301 (Barsamian) Lecture #36 Wed Dec 6, 2023

Pick Up Graded Work

Sign In

Today Finish Section 5.5 Substitution

Friday Review Day

Thurs Dec 14 Final Exam 2:30-4:30 pm

Comprehensive (Covers Everything)

Be sure to review famous values for famous functions

trig functions, e^x, $\ln(x)$

Know rules derivative rules

indefinite integral rules

Know the theorems Intermediate Value, Mean Value, Rolle's, Squeeze

Know how to do Newton's Method

Meeting Part 1: More Examples of Indefinite Integrals Involving Substitution

[Example 1] (Similar to 5.5 #13) Find $\int \frac{1}{7-13x}\,dx$

Solution For clarity, put in parentheses around inside function.

$$\int \frac{1}{(7-13x)}\,dx$$

Step 1 Identify inner function $\quad 7-13x = u$

Step 2 Build $\quad dx = \frac{1}{u'}\,du \qquad dx = -\frac{1}{13}\,du$

Step 3 Sub, Cancel, Simplify

$$\int \frac{1}{(7-13x)}\,dx \underset{\text{sub}}{\overset{\uparrow}{=}} \int \frac{1}{u}\left(-\frac{1}{13}\,du\right) \underset{\text{simplify}}{\overset{\uparrow}{=}} -\frac{1}{13}\int \frac{1}{u}\,du$$

Step 4 Integrate

$$-\frac{1}{13}\int \frac{1}{u}\,du = -\frac{1}{13}\ln|u| + C$$

sub back

Step 5 Sub Back and Conclusion

$$\boxed{\int \frac{1}{7-13x}\,dx = -\frac{1}{13}\ln|7-13x| + C}$$

[Example 2] (Similar to 5.5 #27) Find $\int \csc^3(x)\cot(x)\,dx$

Solution Add parentheses for clarity $\int (\csc(x))^3 \cot(x)\,dx$

Step 1 $\csc(x) = u$

Step 2 Need to build $dx = \frac{1}{u'}\,du$. Need to get u'

$$u' = \frac{d}{dx}u = \frac{d}{dx}\csc(x)$$

$$\frac{d}{dx}\csc(x) = \frac{d}{dx}\frac{1}{\sin(x)} = \frac{d}{dx}(\sin(x))^{-1} \underset{\text{Chain Rule}}{=} (-1)(\sin(x))^{-1-1}\cdot\cos(x) = -(\sin(x))^{-2}\cdot\cos(x)$$

$$= -\frac{1}{(\sin(x))^2}\cdot\cos(x) = -\frac{\cos(x)}{\sin(x)\sin(x)} = -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$

So $dx = -\frac{1}{\csc(x)\cot(x)}\,du$

Step 3 Sub, cancel Simplify

$$\int (\csc(x))^3 \cot(x)\, dx \underset{\text{sub}}{=} \int u^3 \cot(x) \left(-\frac{1}{\csc(x)\cot(x)} du \right)$$

$$\underset{\text{cancel}}{=} -\int \frac{u^3}{\csc(x)} du$$

$$\underset{\text{sub}}{=} -\int \frac{u^3}{u} du$$

$$\underset{\text{cancel}}{=} -\int u^2 du$$

Step 4 Integrate

$$-\int u^{2} du = -\frac{u^{2+1}}{2+1} + C = -\frac{u^3}{3} + C$$

$n=2$

sub back

Step 5 Sub back + conclusion

$$\boxed{\int \csc^3(x)\cot(x) = -\frac{(\csc(x))^3}{3} + C}$$

Alternate Solution to Example 2

Rewrite original integral in a different way

$$\int \csc^3(x)\cot(x)\,dx \underset{\text{Trick}}{=} \int (\csc(x))^2 \cdot \csc(x)\cot(x)\,dx$$

Step 1 $\csc(x) = u$

Step 2 $dx = -\frac{1}{\csc(x)\cot(x)}\,du$

Step 3

$$\int (\csc(x))^2 \csc(x)\cot(x)\,dx \underset{\text{sub}}{=} \int u^2 \csc(x)\cot(x)\left(-\frac{1}{\csc(x)\cot(x)}\,du\right)$$

$$\underset{\text{cancel}}{=} -\int u^2\,du$$

Rest of problem goes just as it did in previous solution

Meeting Part 2 Definite Integrals Involving Substitution

Book mentions two methods of solution but only gives examples using 2^{nd} method.

I will present the 1^{st} method because I like it better, and to fill the gap in book.

Then, I will also present the 2^{nd} method (in these online notes only), so that you can see the difference

[Example 3] Find $\int_0^1 \sqrt[3]{1+26x}\,dx$

Solution Method 1 Don't convert endpoints

$$\int_{x=0}^{x=1} \sqrt[3]{1+26x}\,dx \underset{\text{FTC 2}}{=} \left(\int \sqrt[3]{1+26x}\,dx\right)\Bigg|_{x=0}^{x=1}$$

indefinite integral details next page

$$= \left(\frac{3}{26(4)}(1+26x)^{4/3}\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{3}{26(4)}\left[(1+26(1))^{4/3} - (1+26(0))^{4/3}\right]$$

$$= \frac{3}{26(4)}\left[27^{4/3} - 1^{4/3}\right]$$

$27^{\frac{4}{3}} = (27^{1/3})^4 = 3^4 = 81$

$$= \frac{3}{26(4)}\left[81-1\right] = \frac{3(\cancel{80})^{20}}{26\cancel{(4)}} = \frac{3 \cdot 10}{13} = \frac{30}{13}$$

Final answer

Indefinite Integral Details

$$\int \sqrt[3]{1+26x}\,dx = \int (1+26x)^{1/3}\,dx$$

use substitution

Step 1 $\quad 1+26x = u$

Step 2 $\quad dx = \frac{1}{26}\,du$

Step 3 Sub, cancel Simplify

$$\int (1+26x)^{1/3}\,dx \underset{\text{Sub}}{=} \int u^{1/3}\left(\frac{1}{26}du\right) \underset{\text{Simplify}}{=} \frac{1}{26}\int u^{1/3}\,du$$

Step 4 Integrate

$$\frac{1}{26}\int u^{1/3}\,du = \frac{1}{26}\,\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{1}{26}\,\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{26(4)}u^{\frac{4}{3}} + C$$

$n = 1/3$

Step 5 Sub back

$$\int \sqrt[3]{1+26x}\,dx = \frac{3}{26(4)}(1+26x)^{\frac{4}{3}} + C$$

Do [Example 3] again, this time using Method 2. Convert Endpoints to u

Goal is to find $\int_{x=0}^{x=1} \sqrt[3]{1+26x}\,dx = \int_{x=0}^{x=1} (1+26x)^{1/3}\,dx$

Step 1 identify inner function $1+26x=u$

Step 2 build $dx = \frac{1}{u'}du$ $dx = \frac{1}{26}du$

New Step 3 Convert the endpoints to u

bottom endpoint: when $x=0$, the value of u is $u=1+26(0)=1$

top endpoint: when $x=1$, the value of u is $u=1+26(1)=27$

Step 4 Sub everything, Cancel, simplify

$$\int_{x=0}^{x=1} (1+26x)^{1/3}\,dx = \int_{u=1}^{u=27} u^{1/3}\,\frac{1}{26}du = \frac{1}{26}\int_{u=1}^{u=27} u^{1/3}\,du$$

sub everything including the endpoints

Simplify

Notice that the endpoints are now values of u, not values of x.

Step 5 Integrate

$$\frac{1}{26}\int_{u=1}^{u=27} u^{1/3}\,du \underset{\uparrow \text{ FTC2}}{=} \frac{1}{26}\left(\int u^{1/3}du\right)\Bigg|_{u=1}^{u=27} = \frac{1}{26}\left(\frac{u^{4/3}}{\frac{4}{3}}\right)\Bigg|_{u=1}^{u=27} = \frac{3}{26(4)}u^{4/3}\Bigg|_{u=1}^{u=27}$$

$$= \frac{3}{26(4)}\left[(27)^{4/3}-(1)^{4/3}\right] \underset{\uparrow}{=} \frac{3}{26(4)}\left[81-1\right]$$

because $27^{4/3} = 27^{\frac{1}{3}\cdot 4} = \left(27^{\frac{1}{3}}\right)^4 = 3^4 = 81$

$$= \frac{3}{26\cancel{(4)}}[\cancel{80}] = \frac{3\cdot\cancel{20}}{\cancel{26}} = \frac{3\cdot 10}{13} = \frac{30}{13}$$ Final answer

Observations: (1) Method 2 has slightly different steps

- There is a new step about converting endpoints to u.
- There is no "substitute back" step.

(So there are still 5 steps, but they are slightly different 5 than the 5 steps for the Substitution method for indefinite integrals.)

(2) methods give same answer and involve same amount of work. I prefer method 1 because I don't have to remember a different set of steps.

Example 4 Find average value of $xe^{(-x^2)}$ on the interval $[0,3]$

Solution We have to compute h

$$h = \frac{1}{3-0}\int_{x=0}^{x=3} x e^{(-x^2)}\,dx$$

$$\underset{\text{FTC 2}}{=} \frac{1}{3}\left(\underbrace{\int x e^{(-x^2)}\,dx}\right)\Bigg|_{x=0}^{x=3}$$

Indefinite Integral Details on later page

$$= \frac{1}{3}\left(-\frac{e^{(-x^2)}}{2} + C\right)\Bigg|_{x=0}^{x=3}$$

$$= \frac{1}{3}\left[\left(-\frac{e^{-(3)^2}}{2} + C\right) - \left(-\frac{e^{-(0)^2}}{2} + C\right)\right]$$

$$= \frac{1}{3}\left(-\frac{e^{-9}}{2} + \frac{e^{0}}{2}\right)$$

factor the $\frac{1}{2}$ out front

$$= \frac{1}{6}\left(-e^{-9} + 1\right)$$

$$= \frac{1}{6}\left(1 - e^{-9}\right)$$

$$= \boxed{\frac{1}{6}\left(1 - \frac{1}{e^{9}}\right) \quad \text{Final Answer}}$$

Indefinite Integral Details

Need to find $\int x e^{(-x^2)} dx$

$-x^2 = u$

Step 1 Identify Inner Function

$dx = -\frac{1}{2x} du$

Step 2 Build $dx = \frac{1}{u'} du$

Step 3 Sub, Cancel, Simplify

$$\int x e^{(-x^2)} dx \underset{\text{sub}}{=} \int x e^{(u)} \left(-\frac{1}{2x} du\right) \underset{\text{cancel}}{=} \int e^u \left(-\frac{1}{2}\right) du \underset{\text{simplify}}{=} \left(-\frac{1}{2}\right) \int e^u du$$

Step 4 Integrate $\left(-\frac{1}{2}\right) \int e^u du = \left(-\frac{1}{2}\right) e^u + C$

Step 5 Sub Back + Conclusion

$$\boxed{\int x e^{-x^2} dx = \frac{-e^{-x^2}}{2} + C}$$

Do [Example 4] again, this time using Method 2: Convert Endpoints to u

We need to find the value of $h = \frac{1}{3}\int_{x=0}^{x=3} x e^{-x^2}\,dx$

Step 1 Identify Inner Function $\quad -x^2 = u$

Step 2 Build $dx = \frac{1}{u'}\,du$ $\quad dx = -\frac{1}{2x}\,du$

New Step 3 Convert Endpoints to u.

Bottom endpoint $x = 0 \Longrightarrow u = -x^2 = -(0)^2 = 0$

Top endpoint $x = 3 \Longrightarrow u = -(3)^2 = -9$

Step 4 Sub Everything, Cancel, Simplify

$$h = \frac{1}{3}\int_{x=0}^{x=3} x e^{-x^2}\,dx = \frac{1}{3}\int_{u=0}^{u=-9} x e^{u}\left(-\frac{1}{2x}\,du\right) = \frac{1}{3}\int_{u=0}^{u=-9} e^{u}\left(-\frac{1}{2}\right)du = -\frac{1}{6}\int_{u=0}^{u=-9} e^{u}\,du$$

Sub everything including the endpoints

Cancel

Simplify

Step 5 Integrate

$$h = -\frac{1}{6}\int_{u=0}^{u=-9} e^u\,du \underset{\text{FTC2}}{=} -\frac{1}{6}\left(\int e^u\,du\right)\Bigg|_{u=0}^{u=-9} = -\frac{1}{6}\left(e^u + C\right)\Bigg|_{u=0}^{u=-9}$$

$$= -\frac{1}{6}\left[\left(e^{-9} + \cancel{C}\right) - \left(e^0 + \cancel{C}\right)\right] = -\frac{1}{6}\left[\frac{1}{e^9} - 1\right]$$

$$= \boxed{\frac{1}{6}\left[1 - \frac{1}{e^9}\right] \quad \text{Final Answer}}$$

Observations:
- Methods 1 + 2 gave same answer
- Method 2 seemed a little simpler for this Example

End of 2nd Solution to Example 4

End of Lecture