Information about Limits from Stewart Section 1.4

Limit Laws

Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$ when n is any positive integer.
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ when n is any positive integer. (With the additional requirement that if n is even, then a must be positive. That is, $a > 0$.)
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ when n is any positive integer. (With the additional requirement that if n is even, then $\lim_{x \to a} f(x)$must be positive. That is, $\lim_{x \to a} f(x) > 0$.)

Direct Substitution Property

If f is a polynomial, or if f is a rational function with the number a in its domain, then

$$\lim_{x \to a} f(x) = f(a)$$

Missing Theorem About Limits of Ratios

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, with $\lim_{x \to a} f(x) \neq 0$ and $\lim_{x \to a} g(x) = 0$,

then the ordinary limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ *does not exist* (using the **ordinary definition of limits**).

(This situation gets revisited in Section 1.6 when we study **limits involving infinity**.)

Missing Terminology About Certain Kinds of Limits of Ratios:

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then the $\lim_{x \to a} \frac{f(x)}{g(x)}$ is said to be in $\frac{\mathbf{0}}{\mathbf{0}}$ ***indeterminate form***.

It is not possible to say whether the limit exists, and what its value might be, without first doing some analysis. We'll be learning various techniques in Section 1.4 and Section 1.6.

Useful Fact

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ match.

That is, either $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, or both limits do not exist.

Theorem [2] (The One-Sided Limit Test)

The limit $\lim_{x \to a} f(x) = L$ if and only if f passes this three-part test.

1. The left limit $\lim_{x \to a^-} f(x)$ exists.
2. The right limit $\lim_{x \to a^+} f(x)$ exists.
3. The limits in (1) and (2) match, with common value L. That is, $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Theorem [3] About Limits of Two Functions Where One Function Is Bounded by The Other

If $f(x) \leq g(x)$ when x is near a (except possibly at a), and if the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Theorem [4] The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

then

$$\lim_{x \to a} g(x) = L$$

Equation [6]

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$