Incidence Axiom 1. For every pair of distinct points, there exists exactly one line that both points lie on.

Incidence Axiom 2. For every line there exist at least two distinct points that both lie on the line.

Incidence Axiom 3. There exist three points that do not all lie on any one line.

Incidence Theorem 2.6.2. In incidence geometry, lines that are not parallel intersect in exactly one point.

Incidence Theorem 2.6.3. In incidence geometry, for every line, there exists at least one point that does not lie on the line.

Incidence Theorem 2.6.4. In incidence geometry, for every point, there exist at least two distinct lines that the point lies on.

Incidence Theorem 2.6.5. In incidence geometry, for every line l, there exist lines m and n such that l, m, n are distinct and both m and n intersect l.

Incidence Theorem 2.6.6. In incidence geometry, for every point, there exists at least one line that the point does not lie on.

Incidence Theorem 2.6.7. In incidence geometry, there exist three distinct lines such that no point lies on all three of the lines.

Incidence Theorem 2.6.8. In incidence geometry, for every point P, there exist points Q and R such that P, Q, R are noncollinear.

Incidence Theorem 2.6.9. In incidence geometry, for every pair of distinct points P, Q, there exists a point R such that P, Q, R are noncollinear.

3.1.1 Neutral Axiom (The Existence Postulate). There exist at least two points.

3.1.2 Neutral Definition of the Plane The set of all points is called *the plane* and is denoted by $\mathbb{P}$.

3.1.3 Neutral Axiom (The Incidence Postulate). Lines are sets of points. For every pair of distinct points, there exists exactly one line that both points lie on.

3.1.4 Neutral Definition of *Lies On*

words: Point P *lies on* line l

alternate words: Point P is incident with line l.

alternate words: Line l is incident with point P.

meaning: Point P is an element of line l, denoted $P \in l$

3.1.5 Neutral Definition

words: Point Q is an *external point* for line l

meaning: Point Q does not lie on l. That is, $Q \notin l$.

3.1.5 Neutral Definition

words: lines l and m are *parallel*

symbol: $l \parallel m$

meaning: The interesection of l and m is the empty set, denoted $l \cap m = \phi$. That is, there is no point P such that P lies on both l and m.

3.1.7 Neutral Theorem In neutral geometry, if two distinct lines intersect, then they intersect exactly once. That is, there exists exactly one point that lies on both of the lines.

3.2.1 Neutral Axiom (The Ruler Postulate). For every pair of points P and Q (not necessarily distinct), there exists a real number, denoted PQ and called the distance from P to Q. For everly line l, there is a one-to-one correspondence from l to $\mathbb{R}$ such that if P and Q are points the line that correspond to the real numbers x and y, respectively, then $PQ = |x - y|$.

3.2.1 Neutral Axiom (The Ruler Postulate, Mark's Version).

(i) There exists a function $d: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ called the distance function. For a pair of points $(P, Q) \in \mathbb{P} \times \mathbb{P}$, the real number $d(P, Q)$ is called the distance from P to Q, and is sometimes denoted PQ.

(ii) For every line L, there exists at least one bijective function $f: L \to \mathbb{R}$ that obeys the following equation, called the ruler equation.

$$\forall P, Q \in L(d(P, Q) = |f(P) - f(Q)|)$$

3.2.2 Neutral Definition of Collinear and Noncollinear

words: Points A, B, C are *collinear*

meaning: there exists a line l that A, B, C lie on.

words: Points A, B, C are *noncollinear*

meaning: there does not exist a line l that A, B, C lie on.

3.2.3 Neutral Definition of Betweenness

words: C is between A and B.

symbol: $A * B * C$

meaning: A, B, C are distinct, collinear points and $AC + CB = AB$.

3.2.4 Neutral Definition of Segments and Rays

usage for all of this definition: A, B are distinct points

words: segment A, B

symbol: $\overline{AB}$

meaning: the set $\overline{AB} = \{A, B\} \cup \{P | A * P * B\}$

words: ray A, B

symbol: $\overrightarrow{AB}$

meaning: the set $\overrightarrow{AB} = \overline{AB} \cup \{Q | A * B * Q\}$

3.2.5 Neutral Definition of the Length of a Line Segment

words: the length of $\overline{AB}$

meaning: the distance from A to B, denoted AB.

There is some subtlety here: We can speak of the distance from A to B, denoted AB, even when A and B are the same point. In that case, $AB = 0$. But there is only a segment $\overline{AB}$ when A and B are distinct points. So the length of a segment will aways be strictly greater than 0, never equal to 0.

additional terminology

words: $\overline{AB}$ is congruent to $\overline{CD}$

symbol: $\overline{AB} \cong \overline{CD}$

meaning: the length of $\overline{AB}$ equals the length of $\overline{CD}$.

3.2.6 Neutral Definition of Endpoints and Interior Points of Segments and Rays

words: the endpoints of $\overline{AB}$ **meaning:** the points A and B.

words: the interior points of $\overline{AB}$ **meaning:** the points P such that $A * P * B$

words: the endpoint of $\overrightarrow{AB}$ **meaning:** the point A.

words: the interior points of $\overrightarrow{AB}$ **meaning:** the points Q such that $A * B * Q$

3.2.7 Neutral Theorem In neutral geometry, for any points P, Q (not necessarily distinct)

1. $PQ = QP$
2. $PQ \geq 0$
3. $PQ = 0$ if and only if P and Q are the same point

3.2.8 Neutral Corollary In neutral geometry, $A * C * B$ if and only if $B * C * A$. That is, the following statements are equivalent. (That is, they are either both true or both false.)

(1) $A * C * B$

(2) $B * C * A$

3.2.10 The Euclidean metric

In the Cartesian plane model of neutral geometry, the function d defined by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

qualifies as a distance function. (But d is not a distance function for a general neutral geometry!)

3.2.11 The taxicab metric

In the Cartesian plane model of neutral geometry, the function ρ defined by

$$\rho\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

qualifies a distance function. (But ρ is not a distance function for any general neutral geometry!)

3.2.13 Neutral Definition (coordinate functions and coordinates)

words: a *coordinate function* for line l

usage: l is a line in a neutral geometry that has distance function d.

meaning: a bijective function $f: l \to \mathbb{R}$ that obeys the following equation, called the ruler equation.

$$\forall P, Q \in l(d(P, Q) = |f(P) - f(Q)|)$$

additional terminology: For a point $P \in l$, the real number $f(P)$ is called the *coordinate* of P.

3.2.14 Example: Coordinate Functions in the Euclidean metric for the Cartesian plane.

Suppose that l is a line in the Cartesian plane with the Euclidean metric.

If l is non-vertical, with equation $y = mx + b$, then the function $f: l \to \mathbb{R}$ defined by

$$f(x, y) = x\sqrt{1 + m^2}$$

satisfies the requirements to be called a coordinate function for line l.

If l is vertical, with equation $x = a$, then the function $f: l \to \mathbb{R}$ defined by

$$f(x, y) = y$$

satisfies the requirements to be called a coordinate function for line l.

3.2.15 Example: Coordinate Functions in the taxicab metric for the Cartesian plane.

Suppose that l is a line in the Cartesian plane with the Euclidean metric.

If l is non-vertical, with equation $y = mx + b$, then the function $f: l \to \mathbb{R}$ defined by

$$f(x, y) = x(1 + |m|)$$

satisfies the requirements to be called a coordinate function for line l.

If l is vertical, with equation $x = a$, then the function $f: l \to \mathbb{R}$ defined by

$$f(x, y) = y$$

satisfies the requirements to be called a coordinate function for line l.

3.2.16 Neutral Theorem (The Ruler Placement Theorem). In neutral geometry, for any pair of distinct points P, Q, there exists a coordinate function f for line $\overleftrightarrow{PQ}$ such that $f(P) = 0$ and $f(Q) > 0$.

Remark: Mark calls such a coordinate function a ***special coordinate function*** for line $\overleftrightarrow{PQ}$

3.2.17 Neutral Theorem (Betweenness Theorem for Points)(Betweenness of points is related to the ordering of the coordinates of the points). In neutral geometry, for any line l with coordinate function f, and any points A, B, C on line l, the following statements are equivalent. (That is, they are either both true or both false.)

(1) $A * C * B$

(2) $f(A) < f(C) < f(B)$ or $f(A) > f(C) > f(B)$

3.2.18 Neutral Corollary In neutral geometry, for pair of distinct points A, C and any point B on ray $\overrightarrow{AC}$, , the following statements are equivalent. (That is, they are either both true or both false.)

(1) $A * B * C$

(2) $AB < AC$

3.2.19 Neutral Corollary In neutral geometry, for triple of distinct collinear points, exactly one of the points lies between the other two.

3.2.20 Neutral Corollary (Coordinate description of rays) In neutral geometry, for any pair of distinct points A, B, if f is a ***special coordinate function*** for line $\overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$, then

$$\overrightarrow{AB} = \left\{P \in \overleftrightarrow{AB} | f(P) \geq 0\right\}$$

Remarks:

(1) We know that a ***special coordinate function*** for line $\overleftrightarrow{AB}$ exists by the **Ruler Placement Theorem**.

(2) A cleaner presentation of the statement about the sets is

$$\overrightarrow{AB} = f^{-1}([0, \infty))$$

3.2.21 Neutral Definition of Midpoint

words: a midpoint of $\overline{AB}$

meaning: a point M such that $A * M * B$ and $AM = MB$.

3.2.22 Neutral Theorem (Existence and Uniqueness of Midpoints) In neutral geometry, every segment has exactly one midpoint.

3.2.23 Neutral Corollary (Point Construction Theorem) In neutral geometry, for every pair of distinct points A, B and every real number $d \geq 0$, there exists exactly one point C on $\overrightarrow{AB}$ such that $AC = d$.

3.2.25 Neutral Definition of Circle

words: The circle with center O and radius r

symbol: $circle(O, r)$

usage: O is a point in a neutral geometry, and r is a positive real number.

meaning: the set $circle(O, r) = \{P \in \mathbb{P} | OP = r\}$

3.3.1 Neutral Definition of Convex

words: Set S is convex

usage: S is a set of points in a neutral geometry

meaning: For all pairs of distinct points $A, B \in S$, the segment $\overline{AB} \subset S$.

3.3.2 Neutral Axiom (Plane Separation Postulate) In neutral geometry, for every line l, there are two associated sets of points, denoted H_1 and H_2 and called half planes bounded by l, such that

(0) L, H_1, H_2 is a partition of the set of all points

(1) Each of the half planes is a convex set.

(2) If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ}$ intersects line l.

Remark: PSP (1) and PSP(2) are *conditional statements*, so their *contrapositives* are *automatically true*. That is, given a line l, and distinct points P, Q not on l

PSP (1): If P, Q are in the same *half plane*, then $\overline{PQ}$ does not intersect line l.

PSP (1) (contrapositive): If $\overline{PQ}$ does intersect line l, then P, Q are *not* in the same *half plane*.

PSP (2) If P, Q are not in the same *half plane*, then $\overline{PQ}$ intersects line l.

PSP (2) (contrapositive) If $\overline{PQ}$ does not intersect line l, then P, Q are in the same *half plane*.

Mark's More Descriptive Half-Plane Notation

symbol: $H_{\overleftrightarrow{AB},C}$

usage: A, B, C are distinct, non-collinear points

meaning: The half plane of line $\overleftrightarrow{AB}$ containing C

3.3.5 Neutral Definition of Opposite Rays

words: opposite rays

meaning: Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that $B * A * C$

3.3.6 Neutral Definition of Angle

words: angle B, A, C

symbol: $\angle BAC$

usage: rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are non-opposite rays

meaning: The set $\angle BAC = \overrightarrow{AB} \cup \overrightarrow{AC}$.

additional terminology:

- Point A is called the vertex of the angle.
- Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called the sides of the angle.

3.3.7 Neutral Definition of Interior of an Angle

words: interior of angle B, A, C

symbol: $interior(\angle BAC)$

meaning:

If rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are distinct (and non-opposite) rays, then the interior of the angle is defined to be the intersection of two half planes: $interior(\angle BAC) = H_{\overleftrightarrow{AC},B} \cap H_{\overleftrightarrow{AB},C}$

If rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the same ray, then the interior of the angle is defined to be the empty set.

3.3.8 Neutral Definition of Betweenness for Rays

words: ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

meaning: $D \in interior(\angle BAC)$.

3.3.9 Neutral Theorem (The Ray Theorem) In neutral geometry, if l is a line and $A \in l$ and $B \notin l$ and $C \in \overrightarrow{AB}$ and $C \neq A$, then B and C are in the same half plane of l.

3.3.10 Neutral Theorem In neutral geometry, for all noncollinear points A, B, C and all $D \in \overleftrightarrow{BC}$, the following statements are equivalent. (That is, they are either both true or both false.)

(1) $B * D * C$

(2) Ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$

3.3.11 Neutral Definition of triangle

words: triangle A, B, C

symbol: ΔABC

usage: A, B, C are noncollinear points

meaning: The set $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$.

additional terminology:

- Points A, B, C are called the vertices of the angle.
- Segments $\overline{AB}, \overline{BC}, \overline{CA}$ are called the sides of the triangle.

3.3.12 Neutral Theorem (Pasch's Theorem) In neutral geometry, for any triangle ΔABC, if line l intersects side $\overline{AB}$ at a point that is not a vertex, then l intersects side $\overline{AC}$ or $\overline{BC}$ (or both).

3.4.1 Neutral Axiom (Protractor Postulate) In neutral geometry, for any angle $\angle BAC$, there is a real number, denoted $\mu(\angle ABC)$ and called the measure of $\angle ABC$, such that the following conditions are true

(1) $0 \leq \mu(\angle ABC) < 180$

(2) $\mu(\angle ABC) = 0$ if and only if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the same ray

(3) (Angle Construction Postulate) For each real number r such that $0 < r < 180$ and for each half plane H bounded by $\overleftrightarrow{AB}$, there exists a unique ray $\overrightarrow{AE}$ such that $E \in H$ and $\mu(\angle BAE) = r$

(4) (Angle Addition Postulate) If $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then

$$\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)$$

3.4.2 Neutral Definition of Congruence for Angles

words: angles $\angle BAC$ and $\angle EDF$ are congruent

symbol: $\angle BAC \cong \angle EDF$

meaning: $\mu(\angle BAC) = \mu(\angle EDF)$.

3.4.3 Neutral Definitions of Right, Acute, Obtuse Angles

words: $\angle BAC$ is a right angle **meaning:** $\mu(\angle BAC) = 90$.

words: $\angle BAC$ is an acute angle **meaning:** $\mu(\angle BAC) < 90$.

words: $\angle BAC$ is an obtuse angle **meaning:** $\mu(\angle BAC) > 90$.

3.4.4 Neutral Lemma In neutral geometry, if A, B, C, D are distinct points and C, D are in the same half plane of $\overleftrightarrow{AB}$ and D is not on $\overrightarrow{AC}$, then either $C \in interior(\angle BAD)$ or $D \in interior(\angle BAC)$. (Exclusive or.)

3.4.5 Neutral Theorem (Betweenness Theorem for Rays) In neutral geometry, if A, B, C, D are distinct points and C, D are in the same half plane of $\overleftrightarrow{AB}$ and D, then the following two statements are equivalent.

(i) $\mu(\angle BAD) < \mu(\angle BAC)$

(ii) ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

3.4.6 Neutral Definition of Angle Bisector

words: $\overrightarrow{AD}$ is a bisector of $\angle BAC$

meaning: $D \in interior(\angle BAC)$ and $\mu(\angle BAD) = \mu(\angle DAC)$.

3.4.7 Neutral Theorem (Existence and Uniqueness of Angle Bisectors) In neutral geometry, if A, B, C are noncollinear points, then there exists a unique angle bisector for $\angle BAC$.

3.5.1 Neutral Theorem (The Z Theorem) In neutral geometry, if A, B, D, E are distinct points and B, E lie in different half planes of $\overleftrightarrow{AD}$, then $\overrightarrow{AB} \cap \overrightarrow{DE} = \phi$

3.5.2 Neutral Theorem (The Crossbar Theorem) In neutral geometry, if A, B, C, D are noncollinear points and $D \in interior(\angle BAC)$, then ray $\overrightarrow{AD}$ intersects segment $\overline{BC}$.

3.5.3 Neutral Theorem In neutral geometry, for any distinct noncollinear points A, B, C, D, the following two statements are equivalent.

(i) $D \in interior(\angle BAC)$

(ii) ray $\overrightarrow{AD}$ intersects $interior(\overline{BC})$.

3.5.4 Neutral Definition of Linear Pair

words: Two angles form a linear pair.

meaning: The angles can be labeled $\angle BAD$ and $\angle DAC$ with $B * A * C$ and $D \notin \overleftrightarrow{BC}$

3.5.5 Neutral Theorem (Linear Pair Theorem) In neutral geometry, if two angles form a liear pair, then the sum of the measures of their angles is 180.

3.5.7 Neutral Lemma In neutral geometry, for any A, B, C, D, E such that $B * A * C$, the following two statements are equivalent.

(i) $D \in interior(\angle BAE)$

(ii) $E \in interior(\angle CAD)$

3.5.8 Neutral Definition of Perpendicular Lines

words: lines l and m are perpendicular

symbol: $l \perp m$

meaning: There exists a right angle $\angle BAC \subset l \cup m$.

3.5.9 Neutral Theorem In neutral geometry, for any line l and any point P on l, there exists exactly one line m such that $P \in m$ and $m \perp l$.

3.5.10 Neutral Definition In neutral geometry, a perpendicular bisector of a line segment is a line that contains the midpoint of the segment and that is perpendicular to the line determined by the endpoints of the segment.

3.5.11 Neutral Theorem (Existence and Uniqueness of Perpendicular Bisectors) In neutral geometry, for any two distinct points D, E, there exists exactly one perpendicular bisector for $\overline{DE}$.

3.5.12 Neutral Definition of Vertical Pair (Bowtie Angles)

words: Two angles form a vertical pair (The angles are bowtie angles).

meaning: The angles can be labeled $\angle BAC$ and $\angle DAE$ with $B * A * E$ and $C * A * D$

3.5.13 Neutral Theorem (Vertical Angles Theorem)(Bowtie Angles Theorem) In neutral geometry, if two angles form a vertical pair (that is, they are bowtie angles), then they are congruent.

3.5.14 and 3.5.15 Neutral Lemma and Theorem (The Continuity Axiom) Will be entered later

3.6.1 Neutral Definition of Congruent triangles

words: Triangle 1 and Triangle 2 (not necessarily distinct triangles) are congruent.

meaning: There is a correspondence between the vertices of triangle 1 and the vertices of triangle 2 such that all corresponding parts are congruent.

3.6.3 Neutral Axiom (Side-Angle-Side Postulate, SAS) In neutral geometry if two triangles (not necessarily distinct) have a correspondence of vertices such that two sides and the included angle of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

3.6.5 Neutral Theorem (Isosceles Triangle Theorem)(CS→CA) In neutral geometry, if a triangle has two congruent sides, then the two angles opposite those sides are also congruent.

4.1.1 Neutral Definition of Exterior Angle and Remote Interior Angles

words: Exterior angle of a triangle, and its corresponding remote interior angles.

meaning: The triangle can be labeled ΔABC and the exterior angle can be labeled $\angle ACD$ with D satisfying $B * C * D$. The corresponding remote interior angles are $\angle BAC$ and $\angle ABC$.

4.1.2 Neutral Theorem (Neutral Exterior Angle Theorem) In neutral geometry, the measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.

Restated: In any neutral geometry, given any triangle and any exterior angle for the triangle, the following two statements are true:

(i) The measure of the exterior angle is greater than the measure of the remote interior angle at the remote vertex that lies on one of the rays of the angle.

(ii) The measure of the exterior angle is also greater than the measure of the remote interior that does not lie on one of the rays of the angle.

4.1.3 Neutral Theorem (Existence and Uniqueness of Perpendiculars) In neutral geometry, for every line l and every point P (regardless of whether P lies on l), there exists exactly one line m such that $P \in m$ and $m \perp l$.

Un-numberd Neutral Fact proven in exercise 4.1#1 (a triangle can have only one right angle or obtuse angle) In neutral geometry, if one interior of a triangle is right or obtuse, then the other interior angles are both acute.

4.2.1 Neutral Theorem (Angle-Side-Angle Theorem)(ASA) In neutral geometry if two triangles (not necessarily distinct) have a correspondence of vertices such that two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

4.2.2 Neutral Theorem (Converse of the Isosceles Triangle Theorem)(CA➔CS) In neutral geometry, if a triangle has two congruent angles, then the two sides opposite those sides are also congruent.

4.2.3 Neutral Theorem (Angle-Angle-Side Theorem)(AAS) In neutral geometry if two triangles (not necessarily distinct) have a correspondence of vertices such that two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

4.2.4 Neutral Definition of Right Triangle

words: A triangle is a right triangle

meaning: The triangle has an angle that is a right angle.

remark: By the fact proven in exercise 4.1.1, the other two angles of the triangle must be acute.

additional terminology: The side opposite the right angle is called the hypotenuse. The other two sides are called legs.

4.2.5 Neutral Theorem (Hypotenuse-Leg Theorem)(HL) In neutral geometry if two right triangles (not necessarily distinct) have congruent hypotenuses and a congruent leg, then the triangles are congruent.

4.2.6 Neutral Theorem (Existence of a Point that Creates a Congruent Copy of a Given Triangle)

In neutral geometry given a triangle ΔABC, and a segment $\overline{DE}$ such that $\overline{DE} \cong \overline{AB}$, and a half plane H bounded by $\overleftrightarrow{DE}$, there exists a unique point $F \in H$ such that $\Delta DEF \cong \Delta ABC$.

4.2.7 Neutral Theorem (Side-Side-Side Theorem)(SSS) In neutral geometry if two triangles (not necessarily distinct) have a correspondence of vertices such that the three sides of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

4.3.1 Neutral Theorem (Scalene Inequality)(BS ➔ BA and BA ➔ BS. That is, BABS)

In neutral geometry, in any triangle ΔABC, the following statements are equivalent.

(i) $BA > BC$.

(ii) $\mu(\angle BCA) > \mu(\angle BAC)$

4.3.2 Neutral Theorem (Triangle Inequality for Triangles)

In any neutral geometry, if A, B, C are noncollinear points, then $AC < AB + BC$.

4.3.3 Neutral Theorem (Hinge Theorem)

In any neutral geometry, if ΔABC and ΔDEF are triangles (not necessarily distinct) such that $AB = DE$ and $AC = DF$ and $\mu(\angle BAC) < \mu(\angle EDF)$, then $BC < EF$.

4.3.4 Neutral Theorem (Shortest Segment from a Point to a Line)

In any neutral geometry, for any line l and any point P not on l, if F is the foot of the perpendicular from P to l and R is any point on l that is different from F, then $PF < PR$.

4.3.5 Neutral Definition (Distance from a Point to a Line)

Words: The distance from P to l.

Symbol: $d(P, l)$

Usage: l and P are a line and a point in a neutral geometry

Meaning: $d(P, l)$ is the number defined as follows

- If $P \in l$, then $d(P, l) = 0$.
- If $P \notin l$, then $d(P, l) = PF$, where F is the foot of the perpendicular from P to l.

4.3.6 Neutral Theorem (Pointwise Characterization of Angle Bisector)

In any neutral geometry, given any non-collinear points A, B, C and any point $P \in interior(\angle BAC)$, the following statements are equivalent.

(i) P lies on the angle bisector of $\angle BAC$.

(ii) $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$.

4.3.7 Neutral Theorem (Pointwise Characterization of Perpendicular Bisector)

In any neutral geometry, given any distinct points A, B and any point P, the following statements are equivalent:

(i) P lies on the perpendicular bisector of $\overline{AB}$.

(ii) $PA = PB$.

4.3.8 Neutral Theorem (Continuity of Distance)

In neutral geometry, given

- Non-collinear points A, B, C
- A special coordinate function g for $\overleftrightarrow{AB}$ with the property that $g(A) = 0$ and $g(B) > 0$

Claim: The function: $f_{ABC}: [0, g(B)] \to [0, \infty)$ defined by $f_{ABC}(x) = d\big(C, g^{-1}(x)\big)$ is continuous.

4.4.1, 4.4.3 Neutral Definitions

- Two lines are cut by a transversal
- Interior Angles
- Alternate Interior Angles
- Corresponding Angles
- Nonalternating Interior Angles on the same side of the transversal

See descriptions in the book on pages 82, 83

4.4.2 Neutral Theorem (Alternate Interior Angles Theorem)

In any neutral geometry, given lines l and l' cut by a transversal t,

if a pair of alternate interior angles is congruent, then l and l' are parallel.

4.4.4 Neutral Corollary (Corresponding Angles Theorem)

In any neutral geometry, given lines l and l' cut by a transversal t,

if a pair of corresponding angles is congruent, then l and l' are parallel.

4.4.5 Neutral Corollary (Corresponding Angles Theorem)

In any neutral geometry, given lines l and l' cut by a transversal t, if a pair of nonalternating angles on the same side of the transversal have measures that sum to 180, then $l \parallel l'$.

4.4.6 Neutral Corollary (Existence of Parallels)

In any neutral geometry, given any line l and any point P that does not lie on l, there exists at least one line m such that $P \in m$ and $m \parallel l$.

Important Statement that may or may not be true: the Elliptic Parallel Postulate

Given any line l and any point P that does not lie on l, there is no line m such that $P \in m$ and $m \parallel l$.

4.4.7 Neutral Corollary

In any neutral geometry, the statement of the Elliptic Parallel Postulate is a false statement.

4.4.8 Neutral Corollary

In any neutral geometry, given any lines l, m, n (not necessarily distinct),
if $m \perp l$ and $l \perp n$, then either $m \parallel n$ or m and n are the same line.

4.5.1 Neutral Definition (Angle Sum for a Triangle)

Words: The angle sum for ΔABC.

Symbol: $\sigma(\Delta ABC)$

Meaning: the number $\sigma(\Delta ABC) = \mu(\angle ABC) + \mu(\angle BCA) + \mu(\angle CAB)$

4.5.2 Neutral Theorem (Saccheri-Legendre Theorem)

In any neutral geometry, for any triangle ΔABC, the angle sum is $\sigma(\Delta ABC) \leq 180$.

4.5.3 Neutral Lemma (Used in the proof of the Saccheri-Legendre Theorem)

In any neutral geometry, for any triangle ΔABC, $\mu(\angle CAB) + \mu(\angle ABC) < 180$.

4.5.4 Neutral Lemma (Used in the proof of the Saccheri-Legendre Theorem)

In any neutral geometry, for any triangle ΔABC and any point E such that $B * E * C$,
$\sigma(\Delta ABE) + \sigma(\Delta AEC) = \sigma(\Delta ABC) + 180$.

4.5.5 Neutral Lemma (Used in the proof of the Saccheri-Legendre Theorem)

In any neutral geometry, for any triangle ΔABC, there exists a point $D \notin \overleftrightarrow{AB}$ such that the following two statements are both true

- $\sigma(\Delta ABD) = \sigma(\Delta ABC)$
- $\mu(\angle BAD) \leq \frac{1}{2}\mu(\angle ABC)$ or $\mu(\angle ADB) \leq \frac{1}{2}\mu(\angle ABC)$

4.5.6 Neutral Corollary

In any neutral geometry, for any triangle ΔABC, the sum of the measures of any two interior angles is less than or equal to the sum of their remote exterior angle.

4.5.7 Neutral Corollary (Converse of the Statement of Euclid's Fifth Postulate)

In any neutral geometry, given lines l and l' cut by a transversal t, if lines l and l' intersect on one side of t, then the pair of nonalternating angles on that same side of t have measures that sum to strictly less than 180.

4.6.1 Neutral Definition (Quadrilateral, and associated terminology)

Words: Quadrilateral A, B, C, D.

Symbol: $\square ABCD$

Usage: A, B, C, D are points satisfying the following conditions

No three of the points are collinear.

Any two of the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ either don't intersect, or intersect only at an endpoint.

Meaning: the set of points number $\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$

Additional terminology:

- The four segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are called the **sides** of the quadilateral
- The four points A, B, C, D are called the **vertices** of the quadilateral
- The sides $\overline{AB}, \overline{CD}$ are called **opposite sides**; The sides $\overline{BC}, \overline{DA}$ are also opposite sides.
- The **diagonals** of the quadrilateral are the segments $\overline{AC}, \overline{BD}$
- The **angles** of the quadrilateral are $\angle ABC, \angle BCD, \angle CDA, \angle DAB$
- Two quadrilaterals are said to be **congruent** if there is a correspondence between their vertices so that all pairs of corresponding sides are congruent and all pairs of corresponding angles are congruent.

4.6.2 Neutral Definition (Convex Quadrilateral)

Words: Quadrilateral $\square ABCD$ is convex.

Meaning: each vertex of the quadrilateral is contained in the interior of the angle formed by the other three vertices.

Remark: The book's use of the expression **convex quadrilateral** is **TERRIBLE**. Keep in mind that this new use of the word convex is not the same as our previous use of the word **convex** to describe certain sets of points (Definition 3.3.1 that we learned back in Chapter 3 and found on page 7 of this list). Indeed, by that earlier definition of convex, no quadrilateral would be convex. (Note for instance, that for every quadrilateral $\square ABCD$, the segment $\overline{AC}$ is not a subset of the quadrilateral.) It would have been far better if a new expression such as ***convex shaped quadrilateral*** had been used for this new criterion of each vertex of the quadrilateral being contained in the interior of the angle formed by the other three vertices. Instead, now when you read the word convex, you will have to keep in mind that there are two definition of the word convex in our book, and they are very different from each other.

4.6.3 Neutral Definition (Angle Sum for a Quadrilateral)

Words: The angle sum for $\square ABCD$.

Symbol: $\sigma(\square ABCD)$

Meaning: the number $\sigma(\square ABCD) = \mu(\angle ABC) + \mu(\angle BCD) + \mu(\angle CDA) + \mu(\angle DAB)$

4.6.4 Neutral Theorem

In any neutral geometry, for any quadrilateral $\square ABCD$, the angle sum is $\sigma(\square ABCD) \leq 360$.

4.6.5 Neutral Definition (Parallelogram)

Words: Parallelogram A, B, C, D.

Meaning: A quadrilateral $\square ABCD$ that has the extra property that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$.

4.6.6 Neutral Theorem

In any neutral geometry, for any quadrilateral,

if the quadrilateral is a parallelogram, then the quadrilateral is a convex quadrilateral.

4.6.7 Neutral Theorem

In any neutral geometry, for any ΔABC, if $A * D * B$ and $A * E * C$, then $\square BCED$ is a convex quadrilateral.

4.6.8 Neutral Theorem

In any neutral geometry, for any quadrilateral, the following two statements are equivalent:

(i) The quadrilateral is a convex quadrilateral.

(ii) The diagonal segments intersect.

4.6.9 Neutral Theorem

In any neutral geometry, given points A, B, C, D such that $\square ABCD$ is known to be a quadrilateral, the following two statements are equivalent:

(1) $\square ACBD$ is also a quadrilateral.

(2) $\square ABCD$ is not a convex quadrilateral.

Two important statements that may or may not be true:

Converse of the Statement of the Alternate Interior Angles Theorem

Given lines l and l' cut by a transversal t,

if lines l and l' are parallel, then both pairs of alternate interior angles are congruent.

Euclidean Parallel Postulate

Given any line l and any point P that does not lie on l, there exists exactly one line m such that $P \in m$ and $m \parallel l$.

4.7.1 Neutral Theorem

In any neutral geometry, the following two statements are equivalent. (That is, either both statements are true, or both statements are false.)

(i) Converse of the Statement of the Alternate Interior Angles Theorem

(ii) The Euclidean Parallel Postulate

Another Important Statement that may or may not be true:

Euclid's Postulate V

Given lines l and l' cut by a transversal t, if the sum of the measures of two interior angles on one side of t is less than 180, then l and l' intersect on that side of t.

4.7.2 Neutral Theorem

In any neutral geometry, the following two statements are equivalent. (That is, either both statements are true, or both statements are false.)

(i) Euclid's Postulate V

(ii) The Euclidean Parallel Postulate

4.7.3 Neutral Theorem

In any neutral geometry, each of the following statements is equivalent to the Euclidean Parallel Postulate. That is, either both statements are true, or both statements are false.

(1) (Proclus's Axiom) If l and l' are parallel lines and $t \neq l$ is a line such that t intersects l, then t also intersects l'.

(2) If l and l' are parallel lines and t is a line such that $t \perp l$, then $t \perp l'$.

(3) If k, l, m, n are lines such that $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.

(4) (Transitivity of Parallelism) If $l \parallel m$ and $m \parallel n$, then either $m = n$ or $m \parallel n$.

Another Important Statement that may or may not be true:

Angle Sum Postulate: For any triangle ΔABC, the angle sum is $\sigma(\Delta ABC) = 180$.

4.7.4 Neutral Theorem

In any neutral geometry, the following two statements are equivalent.

(i) The Angle Sum Postulate

(ii) The Euclidean Parallel Postulate

4.7.5 Neutral Lemma (used in the proof of Theorem 4.7.4)

In any neutral geometry, given any right angle $\angle PQQ'$ and given any number $\epsilon > 0$, there exists a point $T \in \overrightarrow{QQ'}$ such that $\mu(\angle PTQ) < \epsilon$.

4.7.6 Neutral Definition (Similar Triangles)

Words: ΔABC and ΔDEF are similar

Meaning: there is a correspondence of vertices of the two triangles such that each pair of corrsponding angles is congruent.

Additional notation

symbol: $\Delta ABC \sim \Delta DEF$

meaning: ΔABC and ΔDEF are similar using the correspondence $(A, B, C) \leftrightarrow (D, E, F)$

Another Important Statement that may or may not be true:

Wallis's Postulate (The existence of triangles that are similar but not congruent)

For any triangle ΔABC and any segment $\overline{DE}$, there exists a point F such that $\Delta ABC \sim \Delta DEF$.

4.7.7 Neutral Theorem: In any neutral geometry, the following two statements are equivalent.

(i) Wallis's Postulate

(ii) The Euclidean Parallel Postulate

4.8.1 Neutral Definition (Defect of a Triangle and Defect of a Quadrilateral)

Words: The defect of ΔABC.

Symbol: $\delta(\Delta ABC)$

Meaning: the number $\delta(\Delta ABC) = 180 - \sigma(\Delta ABC)$

Words: The defect of $\square ABCD$.

Symbol: $\delta(\square ABCD)$

Usage: $\square ABCD$ is a convex quadrilateral

Meaning: the number $\delta(\Delta ABC) = 360 - \sigma(\square ABCD)$

4.8.2 Neutral Theorem (Additivity of Defect):

(1) In any neutral geometry, for any triangle ΔABC and any point E such that $B * E * C$,

$$\delta(\Delta ABC) = \delta(\Delta ABE) + \delta(\Delta AEC)$$

(2) In any neutral geometry, for any convex quadrilateral $\square ABCD$,

$$\delta(\square ABCD) = \delta(\Delta ABC) + \delta(\Delta ACD)$$

4.8.3 Neutral Definition: A **rectangle** is a quadrilateral in which every angle is a right angle.

4.8.4 Neutral Theorem (Six equivalent statements)

In any neutral geometry, the following six statements are equivalent.That is, either all six statements are true, or all six statements are false.

(1) There exists a triangle whose defect is 0.

(2) There exists a right triangle whose defect is zero.

(3) There exists a rectangle.

(4) There exist arbitrarily large rectangles.

(5) The defect of every right triangle is 0.

(6) The defect of every triangle is 0.

4.8.6 Neutral Lemma (used in the proof of Theorem 4.8.4)

In any neutral geometry, in any triangle, at least two of the vertices are acute. Furthermore if ΔABC has acute angles at vertices A and B, then the foot of the perpendicular from vertex C to line $\overleftrightarrow{AB}$ will be a point F such that $A * F * B$.

Another Important Statement that may or may not be true:

Clairaut's Axiom: There exists a rectangle.

4.8.7 Neutral Corollary: In any neutral geometry, the following two statements are equivalent.

(i) Clairaut's Axiom

(ii) The Euclidean Parallel Postulate

4.8.8 Neutral Definition (Saccheri Quadrilateral)

A Saccheri quadrilateral is a quadrilateral $\square ABCD$ such that the angles at vertices A and B are right angles and $\overline{AD} \cong \overline{BC}$.

Additional terminology:

- The angles at vertices A and B are called the base angles.
- The angles at vertices C and D are called the summit angles.
- Segment $\overline{AB}$ is called the base; Segment $\overline{CD}$ is called the summit.

4.8.9 Neutral Definition (Lambert Quadrilateral)

A Lambert quadrilateral is a quadrilateral that has three right angles.

4.8.10 Neutral Theorem (Properties of Saccheri Quadrilaterals)

In any neutral geometry, for any Saccheri quadrilateral $\square ABCD$ with base $\overline{AB}$, the following six statements are all true.

(1) The diagonals are congruent: $\overline{AC} \cong \overline{BD}$.

(2) The summit angles are congruent: $\angle ADC \cong \angle BCD$.

(3) The segment joining the midpoint of $\overline{AB}$ to the midpoint of $\overline{CD}$ is perpendicular to both.

(4) $\square ABCD$ is a parallelogram.

(5) $\square ABCD$ is a convex quadrilateral.

(6) The summit angles are either right or acute.

4.8.11 Neutral Theorem (Properties of Lambert Quadrilaterals)

In any neutral geometry, for any Lambert quadrilateral $\square ABCD$ with right angles at A, B, C, the following four statements are all true.

(1) $\square ABCD$ is a parallelogram.

(2) $\square ABCD$ is a convex quadrilateral.

(3) The angle at vertex D is either right or acute.

(4) $BC \leq AD$

Axioms of Euclidean Geom: The Six Neutral Axioms and the Euclidean Parallel Postulate

3.1.1 Neutral Axiom (Existence). There exist at least two points.

3.1.3 Neutral Axiom (Incidence). Lines are sets of points. For every pair of distinct points, there exists exactly one line that both points lie on.

3.2.1 Neutral Axiom (Ruler, Mark's Version).

(i) There exists a function $d: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ called the distance function. For a pair of points $(P, Q) \in \mathbb{P} \times \mathbb{P}$, the real number $d(P, Q)$ is called the distance from P to Q, and is sometimes denoted PQ.

(ii) For every line L, there exists at least one bijective function $f: L \to \mathbb{R}$ that obeys the following equation, called the ruler equation. $\forall P, Q \in L(d(P, Q) = |f(P) - f(Q)|)$

3.3.2 Neutral Axiom (Plane Separation) In neutral geometry, for every line l, there are two associated sets of points, denoted H_1 and H_2 and called half planes bounded by l, such that

(0) L, H_1, H_2 is a partition of the set of all points

(1) Each of the half planes is a convex set.

(2) If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ}$ intersects line l.

3.4.1 Neutral Axiom (Protractor) In neutral geometry, for any angle $\angle BAC$, there is a real number, denoted $\mu(\angle ABC)$ and called the measure of $\angle ABC$, such that the following conditions are true

(1) $0 \leq \mu(\angle ABC) < 180$

(2) $\mu(\angle ABC) = 0$ if and only if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the same ray

(3) (Angle Construction) For each real number r such that $0 < r < 180$ and for each half plane H bounded by $\overleftrightarrow{AB}$, there exists a unique ray $\overrightarrow{AE}$ such that $E \in H$ and $\mu(\angle BAE) = r$

(4) (Angle Addition) If $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)$

3.6.3 Neutral Axiom (Side-Angle-Side, SAS) In neutral geometry if two triangles (not necessarily distinct) have a correspondence of vertices such that two sides and the included angle of one triangle are congruent to the corresponding parts of the other triangle, then the triangles are congruent.

Euclidean Parallel Postulate (EPP) Given any line l and any point P that does not lie on l, there exists exactly one line m such that $P \in m$ and $m \parallel l$.

5.1.1 Euclidean Theorem (Converse of the Statement of the Neutral Alternate Interior Angles Theorem) In Euclidean geometry, given lines l and l' cut by a transversal t,
if lines l and l' are parallel, then both pairs of alternate interior angles are congruent.

5.1.2 Euclidean Theorem (Euclid's Postulate V)
In Euclidean geometry, given lines l and l' cut by a transversal t,
if the sum of the measures of two interior angles on one side of t is less than 180,
then l and l' intersect on that side of t.

5.1.3 Euclidean Theorem (Angle Sum Postulate)
In Euclidean geometry, for any triangle ΔABC, the angle sum is $\sigma(\Delta ABC) = 180$.

5.1.4 Euclidean Theorem (Wallis's Postulate about the existence of triangles that are similar but not congruent) In Euclidean geometry, for any triangle ΔABC and any segment $\overline{DE}$, there exists a point F such that $\Delta ABC \sim \Delta DEF$.

5.1.5 Euclidean Theorem (Proclus's Axiom) In Euclidean geometry,
If l and l' are parallel lines and $t \neq l$ is a line such that t intersects l, then t also intersects l'.

5.1.6 Euclidean Theorem
In Euclidean geometry, if l and l' are parallel lines and t is a line such that $t \perp l$, then $t \perp l'$.

5.1.7 Euclidean Theorem In Euclidean geometry,
If k, l, m, n are lines such that $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.

5.1.8 Euclidean Theorem (Transitivity of Parallelism)
In Euclidean geometry, if $l \parallel m$ and $m \parallel n$, then either $m = n$ or $m \parallel n$.

5.1.9 Euclidean Theorem (Clairaut's Axiom) In Euclidean geometry, there exists a rectangle.

Facts about Convex Quadrilaterals and Parallelograms in Neutral and Euclidean Geometry

Five Statements that may or may not be true about a convex quadrilateral

Statement a: Each diagonal divides the quadrilateral into congruent triangles.

Statement b: Opposite sides are congruent.

Statement c: Opposite angles are congruent.

Statement d: the diagonals bisect each other

Fifth statement: The quadrilateral is a parallelogram

Exercise 4.6#10 Neutral Theorem (When is a Convex Quad actually a parallelogram?)

In Neutral geometry, given a convex quadrilateral,

(a) If ***Statement a*** is true, then the quadrilateral is a parallelogram.

(b) If ***Statement b*** is true, then the quadrilateral is a parallelogram.

(c) If ***Statement c*** is true, then the quadrilateral is a parallelogram.

(d) If ***Statement d*** is true, then the quadrilateral is a parallelogram..

5.1.10 Euclidean Theorem (Properties of Euclidean Parallelograms)

In Euclidean geometry, given a convex quadrilateral,

(a) If the Quadrilateral is a parallelogram, then ***Statement a*** is true.

(b) If the Quadrilateral is a parallelogram, then ***Statement b*** is true.

(c) If the Quadrilateral is a parallelogram, then ***Statement c*** is true.

(d) If the Quadrilateral is a parallelogram, then ***Statement d*** is true.

5.2.1 Euclidean Theorem (Parallel Projection)

Given: Euclidean lines l, m, n, t, t' and points A, A', B, B', C, C' that have the following properties

- l, m, n are distinct parallel lines
- t is a transversal that cuts lines l, m, n at points A, B, C, respectively.
- t' is a transversal that cuts lines l, m, n at points A', B', C', respectively.
- $A * B * C$

Claim:

$$\frac{AB}{AC} = \frac{A'B'}{A'C'}$$

5.2.2 Euclidean Lemma (Used in the proof of Theorem 5.2.1 about Parallel Projection)

Given: Euclidean lines l, m, n, t, t' and points A, A', B, B', C, C' that have the following properties

- l, m, n are distinct parallel lines
- t is a transversal that cuts lines l, m, n at points A, B, C, respectively.
- t' is a transversal that cuts lines l, m, n at points A', B', C', respectively.
- $A * B * C$
- $\overline{AB} \cong \overline{BC}$

Claim: $\overline{A'B'} \cong \overline{B'C'}$

5.3.1 Euclidean Theorem (Fundamental Theorem on Similar Triangles)

In Euclidean geometry, if $\Delta ABC \sim \Delta DEF$, then

$$\frac{AB}{AC} = \frac{DE}{DF}$$

ratio of lengths of two sides of ΔABC = ratio of lengths of corresponding two sides of ΔDEF

5.3.2 Euclidean Corollary

In Euclidean geometry, if $\Delta ABC \sim \Delta DEF$, then there exists a positive number r such that

$$r = \frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC} = \frac{\text{length of side of } \Delta DEF}{\text{length of corresponding side of } \Delta ABC}$$

5.3.3 Euclidean Theorem In Euclidean geometry, if ΔABC and ΔDEF have the property that $\angle CAB \cong \angle FDE$ and $\frac{AB}{AC} = \frac{DE}{DF}$, then $\Delta ABC \sim \Delta DEF$.

5.3.4 Euclidean Theorem In Euclidean geom, if $\Delta ABC, \Delta DEF$ have the property that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$, then $\Delta ABC \sim \Delta DEF$.

5.4.1 Euclidean Theorem (Pythagorean Theorem)

Without Naming Vertices: In Euclidean geometry, in any right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenues.

With Vertices Named, and restated as a conditional statement:

In Euclidean geometry, given ΔABC with lengths of the sides opposite A, B, C denoted by a, b, c,
If the angle at vertex C is a right angle then $a^2 + b^2 = c^2$.

5.4.2 Terminology and Definitions

An ***altitude line*** for a triangle is a line that passes through one of the vertices of the triangle and that is perpendicular to the line determined by the other two vertices. For instance, in ΔABC, an ***altitude line*** from vertex C would be the line that contains C and is perpendicular to $\overleftrightarrow{AB}$.

Remark: An altitude line for a triangle might not intersect the opposite side of the triangle!

An ***altitude segment*** for a triangle is a segment with one endpoint on a vertex of the triangle, and the other endpoint on the foot of the perpendicular from that vertex to the line determined by the other two vertices. For instance, in ΔABC, an ***altitude segment*** from vertex C would be the segment $\overline{CD}$ where point D is the foot of the perpendicular line from C to $\overleftrightarrow{AB}$.

Remark: An altitude segment for a triangle might not intersect the opposite side of the triangle!

I won't use the terminology of the ***height of a triangle*** in the way that the book uses the term, because that definition is not a good one. (As the book defines it, the height of a triangle would depend on the choice of base. The book does not mention that in its definition of height.)

The ***geometric mean*** of positive numbers x, y is defined to be the number $\sqrt{xy}$.

5.4.3 Euclidean Theorem

Given: Euclidean ΔABC with right angle at C, and with D the foot of the perpendicular from C to $\overleftrightarrow{AB}$. (**Remark:** A result from a Ch. 4 homework exercise shows that since $\angle C$ is the biggest angle of the triangle, we are guaranteed that $A * D * B$.)

Claim: $CD = \sqrt{AD \cdot BD}$. (Note: This calculation is a ***geometric mean***.)

5.4.4 Euclidean Theorem Given: Same stuff as in previous theorem

Claim: $AC = \sqrt{AD \cdot AB}$ and $BC = \sqrt{BD \cdot BA}$ (Note: These calculations are ***geometric means***.)

5.4.5 Euclidean Theorem (Converse of the Statement of Pythagorean Theorem)

In Euclidean geometry, given ΔABC with lengths of the sides opposite A, B, C denoted by a, b, c, If $a^2 + b^2 = c^2$, then the angle at vertex C is a right angle.

5.5.1 Definitions of the Trigonometric Functions in Euclidean Geometry

In Euclidean geometry, when θ is an acute angle $\angle BAC$ such that $\overline{BC} \perp \overline{AC}$

Define $\sin(\theta)$ and $\cos(\theta)$ by the formulas $\sin(\theta) = \frac{BC}{AB} = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos(\theta) = \frac{AC}{AB} = \frac{\text{adjacent}}{\text{hypotenuse}}$

When θ is an obtuse angle in Euclidean geometry

Define $\sin(\theta)$ and $\cos(\theta)$ by the formulas $\sin(\theta) = \sin(\theta')$ and $\cos(\theta) = \cos(\theta')$, where θ' is an angle that forms a linear pair with angle θ

When θ is a straight angle in Euclidean geometry

Define $\sin(\theta)$ and $\cos(\theta)$ by the formulas $\sin(\theta) = 0$ and $\cos(\theta) = 1$.

When θ is a right angle in Euclidean geometry

Define $\sin(\theta)$ and $\cos(\theta)$ by the formulas $\sin(\theta) = 1$ and $\cos(\theta) = 0$.

5.5.2 Euclidean Theorem (Pythagorean Identity)

For any angle θ in Euclidean Geometry, $(\sin(\theta))^2 + (\cos(\theta))^2 = 1$

5.5.3 and 5.5.4 Euclidean Theorems For any triangle ΔABC in Euclidean Geometry

(Theorem 5.5.3 Law of Sines) $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

(Theorem 5.5.4 Law of Cosines) $c^2 = a^2 + b^2 - 2ab\cos(C)$

5.6.2 (Median Concurrence Theorem for Euclidean Triangles)

For any triangle in Euclidean Geometry, the three medians are concurrent. Furthermore, the point of concurrency (POC) has the property that on any one of the medians, the distance from the POC to a triangle vertex is twice the distance from the POC to the other endpoint of the median.
That is, if ΔABC is a triangle in Euclidean geometry, and points D, E, F are the midpoints of the sides opposite A, B, C, then $\overline{AD} \cap \overline{BE} \cap, \overline{CF} = G$ such that $AG = 2GD, BG = 2GE, CG = 2GF$.

Remark: It is possible to prove a Median Concurrence Theorem in neutral geometry, but the proof is much harder and so our book does not present the theorem or its proof.

Definition: The **centroid** of a triangle is the point where the three medians intersect. By Theorem 5.6.2 and the remarks after, we know that every triangle in neutral geometry has a centroid.

Theorem proved in Exercise 5.6#5 (Perpendicular Bisector Concurrence Theorem for Euclidean Triangles) For any triangle in Euclidean Geometry, the perpendicular bisectors of the three sides are concurrent. Furthermore, the point of concurrency (POC) has the property that it is equidistant from the vertices of the triangle.

Remark: There is no Perpendicular Bisector Concurrence Theorem for neutral geometry, because there are examples of triangles in neutral geometry where the perpendicular bisectors of the three sides do not intersect!

Definition: The **circumcenter** of a Euclidean triangle is the point where the perpendicular bisectors of the three sides intersect. By the theorem from Exercise 5.6#5, we know that every Euclidean triangle has a circumcenter. But by the remarks above, we realize that in there are some triangles in neutral geometry that do not have a circumcenter.

Theorem proved in Exercise 5.6#7 (Altitude Line Concurrence Theorem for Euclidean Triangles) For any triangle in Euclidean Geometry, the three altitude lines are concurrent.

Remark: There is no Altitude Line Concurrence Theorem for neutral geometry, because there are examples of triangles in neutral geometry where the three altitude lines do not intersect.

Definition: The **orthocenter** of a Euclidean triangle is the point where the three altitude lines intersect. By the theorem from Exercise 5.6#7, we know that every Euclidean triangle has a orthocenter. But by the remarks above, we realize that in there are some triangles in neutral geometry that do not have an orthocenter.

5.6.3 (Euler Line Theorem for Euclidean Triangles) For any triangle in Euclidean geometry, with orthocenter H, centroid G, and circumcenter O,

- If the triangle is equilateral, then the three points H, G, O are actually the same point.
- If the triangle is not equilateral, then $H * G * O$ and $HG = 2GO$. In this case, the line containing the three points H, G, O is called the ***Euler line*** of the triangle.

Definition of *Cevian Lines* and *Proper Cevian Lines*

- A ***Cevian line*** for a triangle is a line that passes through a vertex of the triangle and through a point on the opposite sideline. (That is, the line that passes through the other two vertices.)
- A Cevian line is called ***proper*** if it passes through only one vertex of the triangle.

5.6.4 (Ceva's Theorem)

Given: Euclidean triangle ΔABC with ***proper Cevian lines*** $\overleftrightarrow{AL}, \overleftrightarrow{BM}, \overleftrightarrow{CN}$ where $L \in \overleftrightarrow{BC}$ and $M \in \overleftrightarrow{CA}$ and $N \in \overleftrightarrow{AB}$

Claim: The following statements are equivalent. That is, they are either both true or both false.

(1) The lines $\overleftrightarrow{AL}, \overleftrightarrow{BM}, \overleftrightarrow{CN}$ are concurrent (or mutually parallel)

(2) $\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$

Definition of *Menelaus Points* and *Proper Menelaus Points*

- A ***Menelaus point*** for a triangle is a point on one of the sidelines of the triangle.
- A Menelaus point is called ***proper*** if it is not one of the vertices of the triangle.

5.6.5 (Theorem of Menelaus)

Given: Euclidean triangle ΔABC with ***proper Menelaus points*** $L \in \overleftrightarrow{BC}$ and $M \in \overleftrightarrow{CA}$ and $N \in \overleftrightarrow{AB}$

Claim: The following statements are equivalent. That is, they are either both true or both false.

(1) The three points L, M, N are collinear

(2) $\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$

Definition of the *Morley Triangle*

In any triangle ΔABC in neutral geometry, there exist two angle trisectors for each of the three angles. These trisectors are rays. Any ray that is a trisector of one of the angles will intersect any ray that is a trisector of one of the other angles at an intersection point that is in the interior of the triangle. The six rays that are the trisectors therefore create twelve points of intersection.
Label three of those twelve intersection points as follows:

- For the trisectors of B and C that are closest to $\overline{BC}$, label the point of intersection as A'.
- For the trisectors of C and A that are closest to $\overline{CA}$, label the point of intersection as B'.
- For the trisectors of A and B that are closest to $\overline{AB}$, label the point of intersection as C'.

Triangle $\Delta A'B'C'$ is called the ***Morley triangle*** for ΔABC.

5.6.5 (Morley's Theorem for Euclidean Geometry)

For any triangle ΔABC in Euclidean Geometry, the associate Morley triangle $\Delta A'B'C'$ is equilateral.

7.1.1 Definition of Triangle Interior

Symbol: $\text{Int}(\Delta ABC)$

Spoken: the *interior* of ΔABC

Meaning: $H_{\overleftrightarrow{AB},C} \cap H_{\overleftrightarrow{BC},A} \cap H_{\overleftrightarrow{CA},B}$

7.1.2 Definition of Triangular Region

Symbol: $\blacktriangle ABC$

Spoken: triangular region A, B, C

Meaning: The set of points $\blacktriangle ABC = \Delta ABC \cup \text{Int}(\Delta ABC)$

7.1.3 Definition of Polygonal Region

A ***polygonal region*** is a subset R of the plane that can be written as the union of a finite number of triangular regions in such a way that if two of the triangular regions intersect, then the intersection is contained in an edge of each.

Definition not presented in the book: The set of all Polygonal Regions

Symbol: $\mathcal{R}$

Meaning: The set of all Polygonal Regions

7.1.4 Definition of Triangulation of a Polygonal Region

A ***triangulation*** of a polygonal region R is a particular collection of triangular regions $T_1, T_2, \ldots, T_n$ satisfying the requirements in the definition of Polygonal Region.

Remark on Polygonal Regions and Triangulations: A polygonal region is definied to be a set of points for which a triangulation exists. But for any polygonal region, there are many triangulations.

7.1.5 Definition of Non-Overlapping Polygonal Regions.

Two polygonal regions are said to be non-overlapping if they either do not intersect or intersect only along the edges of each region.

7.1.6 Neutral Area Axiom (an *area function* exists)

(Recall Mark's use of the symbol $\mathcal{R}$ to denote the set of all polygonal regions.)

In neutral geometry, there exists a function $\alpha: \mathcal{R} \to \mathbb{R}^+$, called the ***area function***, such that

1. (Congruence) If $\Delta ABC \cong \Delta DEF$, then $\alpha(\blacktriangle ABC) = \alpha(\blacktriangle DEF)$
2. (Additivity) If R_1 and R_2 are non-overlapping polygonal regions, then

$$\alpha(R_1 \cup R_2) = \alpha(R_1) + \alpha(R_2)$$

Remarks:

- The symbol $\alpha(\blacktriangle ABC)$ would be spoken *the area of triangular region* A, B, C.
- From now on, we will write $\alpha(\Delta ABC)$, and say *the area of triangle* A, B, C to mean $\alpha(\blacktriangle ABC)$.

7.1.7 Neutral Theorem about subdividing a triangular region into non-overlapping regions

Given: a triangle ΔABC in neutral geometry, and any point $E \in interior(\overline{AC})$,

Claim: $\blacktriangle ABE$ and $\blacktriangle EBC$ are non-overlaping regions and $\blacktriangle ABC = \blacktriangle ABE \cup \blacktriangle EBC$

7.1.8 Neutral Theorem about subdividing a convex quadrilateral

Given: a convex quadrilateral $\square ABCD$ in neutral geometry

Claim: $\blacktriangle ABC \cup \blacktriangle CDA = \blacktriangle DAB \cup \blacktriangle BCD$

7.1.9 Definition of polygonal region determined by a convex quadrilateral in neutral geom

Symbol: $\blacksquare ABCD$

Spoken: polygonal region A, B, C, D

Usage: $\square ABCD$ is a convex quadrilateral in neutral geometry

Meaning: the set of points $\blacksquare ABCD = \blacktriangle ABC \cup \blacktriangle CDA = \blacktriangle DAB \cup \blacktriangle BCD$

Remark: The symbol $\alpha(\blacksquare ABCD)$ would be spoken *the area of polygonal region* A, B, C, D. But from now on, we will write $\alpha(\square ABCD)$, and say *the area of quadrilateral* A, B, C, D to mean $\alpha(\blacksquare ABCD)$.

7.2.1 Euclidean Area Axiom The area of any rectangle $\square ABCD$ is $\alpha(\square ABCD) = AB \cdot CD$.

7.2.2 Definition of base and height in Neutral Geometry

Given a triangle in neutral geometry, a ***base*** of the triangle is a particular ***side*** of the triangle. That is, a base is a ***segment***. For a particular base, the corresponding ***height*** is the ***length*** of the altitude segment from the opposite vertex. For example, in ΔABC, if segment $\overline{AB}$ is chosen as the base, then the length of the base is the number AB, and the corresponding height is the number CD, where D is the foot of the perpendicular from C to $\overleftrightarrow{AB}$.

Unnumbered Euclidean Theorem proven in Exercise 7.2#3

For any triangle in Euclidan geometry, the product

$$\text{length of base} \times \text{height}$$

does not depend on which side of the triangle is chosen as the base.

7.2.3 Euclidean Theorem about Triangle Area

In Euclidean geometry, the area of any triangle is $\frac{1}{2}\text{length of base} \times \text{height}$

7.2.4 Euclidean Theorem about the Area of Similar Triangles

In Euclidean geometry, if $\Delta ABC \sim \Delta DEF$ with $\frac{AB}{DE} = r$, then $\frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = r^2$

7.2.5 Definition of a Square and Square on a Segment

A square is a quadrilateral that is both a rectangle (four right angles) and a rhombus (four congruent sides). (**Remark:** Rectangles (and squares) exist only in Euclidean geometry.)
Given a segment $\overline{AB}$, a **square on $\overline{AB}$** is a square $\square ABED$ that has $\overline{AB}$ as a side.

7.2.6 (Euclid's version of the Pythag. Thm.) For any right triangle in Euclidean geometry, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the legs.

7.2.7 Euclidean Theorem (Variation of the Pythagorean Theorem)

Given: Euclidean ΔABC with right angle at C, and points D, E, F such that $\Delta ABD \sim \Delta BCE \sim \Delta CAF$

Claim: $\alpha(\Delta ABD) = \alpha(\Delta BCE) + \alpha(\Delta CAF)$

8.1.1, 8.1.2, 8.1.3 Neutral Geometry Definition of Circle and Associated Terminology

Symbol: $\mathcal{C}(O,r)$

Spoken: the ***circle*** with ***center*** O and ***radius*** r

Usage: O is a point in a neutral geometry, and r is a positive real number

Meaning: The set of points that are a distance r from O. That is, $\mathcal{C}(O,r) = \{P|OP = r\}$

Associated Terminology:

- A ***chord*** of a circle is a segment whose endpoints lie on the circle
- A ***diameter segment*** of a circle is a chord that contains the center of the circle.
- Two points are called ***antipodal points*** of a circle if they are the endpoints of a diameter segment.
- A point is said to be ***inside*** the circle if its distance from the center is ***less than*** r.
- A point is said to be ***outside*** the circle if its distance from the center is ***greater than*** r.

8.1.4 Neutral theorem: In neutral geometry, a line cannot intersect a circle three times (so the number of possible intersection points of line and a circle is 0, 1, or 2).

8.1.5 and 8.1.6 Neutral Geometry Definition of Tangent and Secant Lines for a Circle

In neutral geometry, a line is said to be ***tangent*** to a circle if the line only intersects the circle once. The point of intersection is called the ***point of tangency***.

A line is said to be a ***secant line*** for a circle in neutral geometry if the line intersects the circle in two distinct points.

8.1.7 Neutral Geometry aboutTangent Lines being Perpendicular to Radial Segment

In Neutral Geometry, tangent lines are perpendicular to the radial segment.

Given: A segment $\overline{OP}$ and a line L passing through point P.

Claim: The following statements are equivalent.

(i) $\overline{OP} \perp L$.

(ii) Line L is tangent to $\mathcal{C}(O,OP)$ at point P. That is, L only intersects $\mathcal{C}(O,OP)$ at point P.

8.1.8 Neutral Geometry Theorem about Points on Tangent Lines being Outside the Circle

In neutral geometry, if a line is tangent to a circle, then every point of the line except the point of tangency is outside the circle.

8.1.9 Neutral Theorem about Perpendicular Bisectors of Chords

Given a chord in a circle in neutral geometry, the perpendicular bisector of the chord passes through the center of the circle.

8.1.10 Neutral Theorem about Points on Secant Lines

Given a chord $\overline{PQ}$ of a circle in neutral geometry, every point in the interior of $\overline{PQ}$ lies inside the circle, and every point in $\overleftrightarrow{PQ}\backslash\overline{PQ}$ (that is, every point that lies on $\overleftrightarrow{PQ}$ but not on $\overline{PQ}$) lies outside the circle.

8.1.11 Neutral Theorem about Elementary Circular Continuity

In neutral geometry, if a line contains a point that is inside a circle and a point that lies outside the circle, then the line is a secant line for the circle (that is, the line intersects the circle twice).

8.1.12 Neutral Corollary Saying that a Circle Can't Fully Enclose a Line.

In neutral geometry, if a line contains a point that is inside a circle, then the line is a secant line for the circle (that is, the line intersects the circle twice).

8.1.14 Neutral Geometry Definition of Tangent Circles

Two circles in neutral geometry are said to be ***tangent*** if they intersect in exactly one point.

8.1.15 Neutral Theorem about Tangent Circles

In Neutral Geometry, if $\mathcal{C}(O_1, r_1)$ and $\mathcal{C}(O_2, r_2)$ are tangent at P, then

- the centers O_1, O_2 are distinct
- O_1, O_2, P are collinear
- the circles share a common tangent at P

8.2.1 Neutral Definition of a Circle Circumscribing a Polygon (generalization of Book's definition)

Given a polygon with vertices $P_1, P_2, \dots, P_k$, we say that a circle ***circumscribes the polygon*** if all of the polygon's vertices $P_1, P_2, \dots, P_k$ lie on the circle. The circle is called the ***circumcircle of the polygon***, and the center of the circle is called the ***circumcenter of the polygon***. (It is important to realize that NOT ALL POLYGONS CAN BE CIRCUMSCRIBED!)

8.2.2 Neutral Theorem about Circumscribed Triangles

Given a triangle in Neutral Geometry, the following are equivalent

(i) the perpendicular bisectors of the sides of the triangle are concurrent

(ii) there is a circle that contains all three vertices of the triangle. That is, the triangle can be circumscribed.

Corollary: The point where the perpendicular bisectors meet in (i) is the center of the circle in (ii) (which is the point called the circumcenter). Furthermore, the circle and circumcenter are unique.

8.2.3 Neutral Theorem about when it is known that every triangle can be circumscribed.

In neutral geometry, the following are equivalent

(i) The Euclidean Parallel Postulate

(ii) Every triangle can be circumscribed.

8.2.5 Corollary In Euclidean geometry, the perp bisectors of the three sides of any triangle are concurrent.

8.2.6 Neutral geometry Corollary

If the Euclidean Parallel Postulate is not true, then there exists a triangle that cannot be circumscribed

8.2.7 Neutral geometry Definition of Circles Inscribed in Polygons.

Given a polygon with vertices $P_1, P_2, \dots, P_k$, we say that a circle is ***inscribed in the polygon*** if all of the polygon's sides $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_{k-1}P_k}, \overline{P_kP_1}$, are tangent to the circle. The center of the circle is called the ***incenter of the polygon***.

8.2.8 Neutral geometry Theorem About Concurrence of Angle Bisectors, and Inscribed Circles

In neutral geometry, the bisectors of the interior angles of any triangle are concurrent at a point that is called the ***incenter***. Because of this, every triangle has a unique inscribed circle.

8.2.9 Neutral Geometry Definition of Polygon See the book

8.2.10 Neutral Geometry Definition of Regular Polygon

A ***regular polygon*** is a polygon with all sides congruent and all interior angles congruent.

8.2.11 Neutral Geometry Definition of Polygon Inscribed in a Circle

These sentences mean the same thing:

- The polygon is ***inscribed in the circle***.
- The circle ***circumscribes the polygon***.
- All of the vertices of the polygon lie on the circle.

8.2.11 Neutral Geometry Theorem about Existence of Regular Polygons Inscribed in a Circle

In Neutral Geometry, given a circle, and a point P on the circle, and an integer $n \geq 3$, there exists a regular polygon with P as one of its vertices, and with n sides, that is inscribed in the circle.

8.3.1 Euclidean Theorem

In Euclidean geometry, given ΔABC, with M the midpoint of $\overline{AB}$,
if $AM = MC$, then $\angle ACB$ is a right angle.

8.3.2 Euclidean Corollary

In Euclidean geometry, if the vertices of a triangle lie on a circle and one of the sides of the triangle is a diameter segment of the circle, then the angle opposite that side of the triangle is a right angle.

8.3.3 Euclidean Theorem (Converse of the Statement of Theorem 8.3.1)

In Euclidean geometry, given ΔABC, with M the midpoint of $\overline{AB}$,
if $\angle ACB$ is a right angle, then $AM = MC$.

8.3.4 Euclidean Corollary

In Euclidean geometry, if $\angle ACB$ is a right angle, then $\overline{AB}$ is a diameter segment for the circle that circumscribes ΔABC.

8.3.5 Euclidean 30-60-90 Theorem

In Euclidean geometry, if a triangle has interior angle measures 30, 60, 90, then the side opposite the 30 degree angle (that is, the shortest side) is half as long as the hypotenuse.

8.3.6 Euclidean Converse of the Statement of the 30-60-90 Theorem

In Euclidean geometry, if a triangle has a leg that is half as long as the hypotenuse, then the interior angles of the triangle have measure 30, 60, 90.

8.3.7 Euclidean Definition of Inscribed Angles and Central Angles for a Circle

- An ***inscribed angle*** for a circle is an angle $\angle PQR$ where P, Q, R are ***distinct*** points on the circle.
- A ***central angle*** for the circle is an angle $\angle POR$, where P, R are ***distinct*** points on the circle and the angle vertex O is the center of the circle
- The ***arc intercepted by an inscribed (or central) angle*** is the set of points of the circle that lie in the interior of the angle.
- **Remark:** The book does have the qualifier ***distinct***, but it is essential.

8.3.8 Euclidean Definition of Corresponding Central Angle for *some* Inscribed Angles

Words: central angle ***corresponding to*** inscribed angle $\angle PQR$.

Usage: $\angle PQR$ is inscribed in a circle $\mathcal{C}(O, r)$ and has the extra property that the vertex of the angle and the center of the circle (that is, points Q and O) lie in the same half plane of line $\overleftrightarrow{PR}$.

Meaning: The corresponding central angle is the central angle $\angle POR$. Alternately, one can say that the corresponding central angle is the central angle that intercepts the same arc as the inscribed angle.

Remark. Not all inscribed angles have a corresponding central angle. If inscribed $\angle PQR$ does not have the extra property described above, then it will not have a corresponding central angle, because there could not be a central angle that intercepts the same arc.

8.3.9 Euclidean Central Angle Theorem

In Euclidean geometry, if an inscribed angle for a circle has a corresponding central angle, then the measure of the inscribed angle is one half the measure of the corresponding central angle.

8.3.9 Euclidean Corollary In Euclidean geometry, if two inscribed angles intercept the same arc, then the angles are congruent.

8.3.12 Euclidean Theorem about a Number Associated to Point not on a Circle.

Given: A circle $\mathcal{C}(B, r)$ in Euclidean geometry and a point O not on the circle.

Claim: If a number y is defined as follows,

Choose a line L that contains O and that intersects the circle at least once.

- If L is a tangent line with point of tangency P, define $y = (OP)^2$
- If L is a secant line intersecting the circle at Q, R, define $y = (OQ)(OR)$

then the value of y does not depend on which line L is used.

Definition 8.3.11 The Power of a Point in Euclidean Geometry

Words: The power of O with respect to the circle $\mathcal{C}(B, r)$

Usage: $\mathcal{C}(B, r)$ is a circle in Euclidean geometry and O is a point not on the circle.

Meaning: The number y defined in **Theorem 8.3.12**.

8.4.1 Euclidean Lemma about Certain Lengths on a Right Triangle

In Euclidean geometry, if ΔABC has a right angle at vertex C, and D is a point such that $A * D * B$ and $AD = AC$, then $CD < CB$.

8.4.2 Euclidean Lemma about a Certain Limit Associated to a Point on a Circle

Given these objects in Euclidean geometry

- a circle $\mathcal{C}(O, r)$
- a point C on the circle
- the line t tangent to the circle at C
- a point B on t that is distinct from point C.

Claim: If for each number x such that $0 < x < \mu(\angle BOC)$, a point P_x is defined to be the point with the following properties

- P_x lies on the circle
- P_x is in the interior of $\angle BOC$
- $\mu(P_x OC) = x$

then $\lim_{x \to 0} CP_x = 0$.

8.4.3 Theorem about a Certain Limit Associated to a Point on a Circle

Given these objects in Euclidean geometry

- a circle $\mathcal{C}(O, r)$
- a point C on the circle
- A choice of half plane determined by the line $\overleftrightarrow{OC}$
- a point O'

Claim: If a function function $f: (0,180) \to [0, \infty)$ is defined in the following way:

For each number x such that $0 < x < 180$, the value of $f(x)$ is defined to be $f(x) = O'P_x$, where P_x is the point with the following properties

- P_x lies on the circle
- P_x is in the chosen half plane of the line $\overleftrightarrow{OC}$
- $\mu(COP_x) = x$

then the function f is continuous.

8.4.4 Euclidean Theorem of the Circular Continuity Principle

If two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have the property that $\mathcal{C}_2$ contains a point that lies inside $\mathcal{C}_1$ and also contains a point that lies outside $\mathcal{C}_1$, then the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$ contains exactly two points.