MATH 3110/5110 (Barsamian) Lecture L02 (Fri Jan 19, 2024)

<u>Reminders</u>

- Homework H1 due Fri Jan 26 (a week from today)
- Quiz Q1 Fri Jan 26
- Read the book! It is <u>excellent</u>, and I generally will not spend lecture time duplicating content that is presented in the book. I'll use lectures to highlight certain topics, and to add more details & examples for certain topics where the book does not have enough.
- There is a PDF of the <u>first</u> <u>4</u> <u>chapters</u> of the book on the Blackboard Site. If you dont have a book yet, this will enable you to get started reading. But you will need your own book before too long (we go way beyond Ch 4 in this course.) You will have to read the book to succeed in this course.

Meeting Part 1 Independence

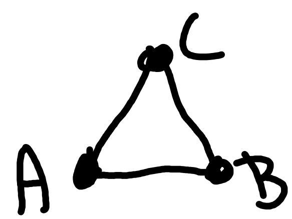
This topic is introduced briefly in Section 2.1 on p.16 with little discussion. But the topic appears throughout later sections. (For instance, the word "independent" is used in Section 2.3 on p.21, and some of the exercises in Section 2.4 are about independence, even if they don't say so.)

Definition

Given some collection of statements $\{S_1, S_2, \dots, S_k\}$, some other statement S is said to be independent of the collection if it is possible for S to be true when all of the $S_1, \dots, S_k$ are true, and it is also possible for S to be false when all of the $S_1, \dots, S_k$ are true. (So it is not possible to prove that S is true (or false) based on the statements $S_1, \dots, S_k$

[Example #1] Involving the Incidence Axioms IA1, IA2, IA3 that are presented at the start of Section 2.2, on page 16.

Consider the collection to be {IA1, IA2}, and investigate possibilities for the truth of IA3.

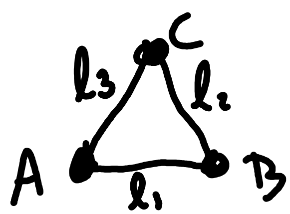
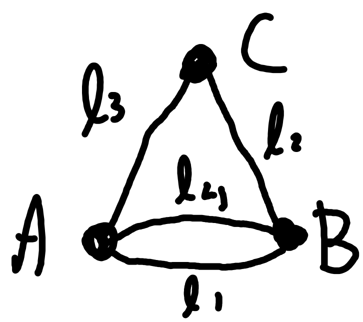
Interpretation		Truth of statements	
Interpretation #1		IA1 IA2 IA3	True True True
Interpretation #2		IA1 IA2 IA3	True True False (because there are not three non-collinear points)

These two interpretations show that IA3 is independent of {IA1, IA2}

End of [Example #1]

[Example #2] Prove that IA1 is independent of {IA2, IA3}

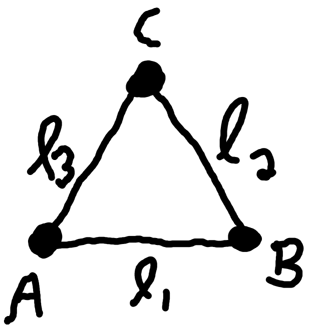
Solution

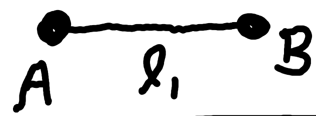
Interpretation	Truth of statements
Interpretation #1 	IA1 True IA2 True IA3 True
Interpretation #3 	IA1 False (Because there are two lines that A, B both lie on.) IA2 True IA3 True

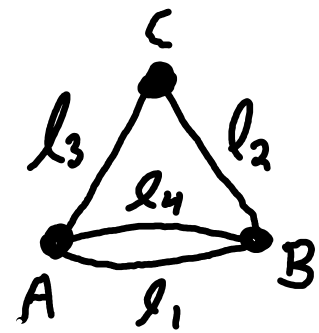
These two interpretations show that IA1 is independent of {IA2, IA3}

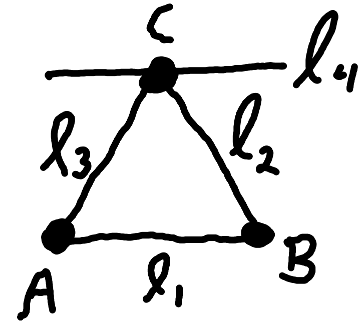
End of [Example #2]

It is possible to make a concise <u>table</u> that can be used to explain Independence of Statements









Truth of Axioms	Interpretations #1	#2	#3	#4
IA1	T	T	F	T
IA2	T	T	T	F
IA3	T	F	T	T

Interpretations #1, #2 show that IA3 is independent of {IA1, IA2}

Interpretations #1, #3 show that IA1 is independent of {IA2, IA3}

Interpretations #1, #4 show that IA2 is independent of {IA1, IA3}

Definition

A whole collection of statements is said to be an independent collection if every statement in the collection is independent of the rest of the collection.

We see that the collection of Axioms for Incidence Geometry, $\{IA1, IA2, IA3\}$ is an independent collection.

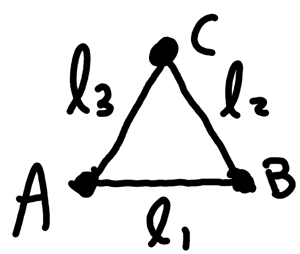
Now consider a new statement S_1

S_1: There are exactly six lines.

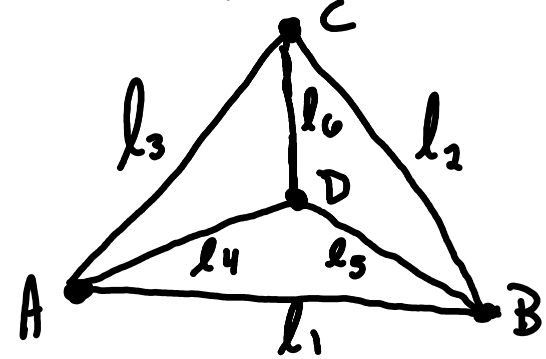
Is S_1 independent of the axioms for Incidence Geometry?

Yes! Consider these two interpretations

Interpretation #1



Interpretation #4



Truth of Statements	Interpretation #1	Interpretation #4
IA1	T	T
IA2	T	T
IA3	T	T
S_1	T	F

Now Consider another new statement, S_2

S_2: For every point P, there are at least two lines that P lies on.

Is statement S_2 independent of the axioms for Incidence Geometry?

No! It is discussed in Section 2.6 that statement S_2 is <u>always</u> <u>true</u> in an Incidence Geometry.

That is, it is possible to <u>prove</u> that if IA1, IA2, IA3 are true, then S_2 will be true as well.

So S_2 is <u>not</u> independent of $\{S_1, S_2, S_3\}$

(In fact, S_2 is a <u>Theorem</u> of Incidence Geometry.)

Finally, Consider one more statement, S_3

S_3: There is a line that all the points lie on.

Is S_3 independent of the axioms of Incidence Geom?

No! If Incidence Axiom IA3 is true, then statement S_3 will have to be false. So S_3 is not independent of {IA1, IA2, IA3}

Meeting Part 2: "Distinct" Objects

I want to bring to your attention something that can be a source of confusion:

<u>Names</u> of Objects

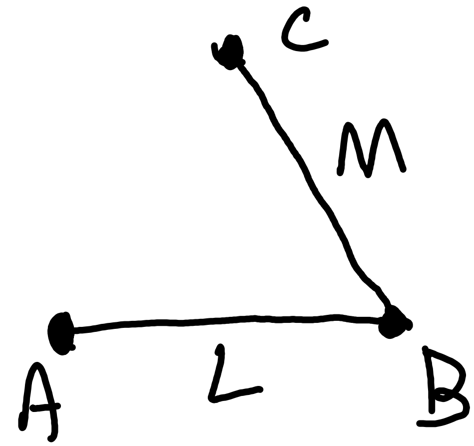
and

"<u>Distinct</u>" objects

We will consider three examples where the symbols L & M are used for lines

[Example #4]

In this example

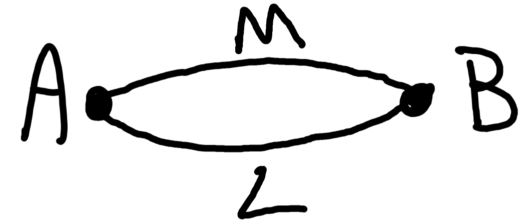


L is a line that has points A & B on it, while M is a line that has points B & C on it.

L & M are not the same line, and they have different names.

No problem

[Example 5]

In this example 

L is a line that has points A & B on it,

and M is also a line that has points A & B on it.

But L & M are not the same line.

(This is analogous to the situation where a traveler has more than one choice of airline flights from New York to chicago.)

So L & M are not the same line, and they have different names. But there is some confusion possible because the two lines have the same points lying on them.

[Example 6]

In this example, L & M are both names for the same line. That might seem impossible, or at least silly. But in fact it happens in the real world all the time.

For example, in New York City, the names

Avenue of the Americas

and

Sixth Avenue

are two names for the same avenue

You might wonder why NYC would do such a thing. 14
More pertinent to us, you might wonder why it would ever happen that some single object could have two different names.

Well it can happen that we can formulate a sequence of logical steps that prove that some object of a certain type exists. We could call that object "L".

And maybe some other set of logical steps would prove that an object of the same type exists. We could call that object "M".

It could be that L & M are <u>different objects</u> (as in example #2 above), or it might turn out that L & M are actually the <u>same object</u> (as in example #6, above).

There are situations like this in Geometry where it is possible to prove that names L & M must actually refer to the <u>same object</u>. But there are also situations that will arise where it will <u>not</u> be possible to prove that L & M refer to the same object, because it might be that they are actually <u>not</u> the same object.

In light of this discussion, I hope you find it understandable that we need to have a definition like the following:

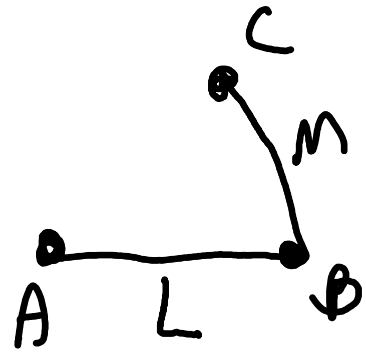
<u>Definition</u>

<u>Words</u> Lines L + M are <u>distinct</u>

<u>Meaning</u>: They are not the same line

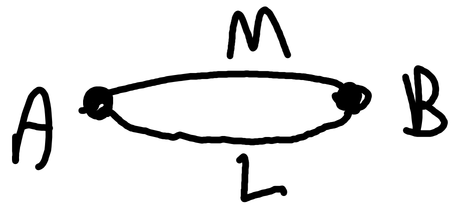
Examples

Example #4



L & M are distinct

Example #5



L & M are distinct

Example #6

L & M are the same line. They are not distinct

And in light of the above discussion, I think that you can understand the need to add the qualifier "distinct" when you mean to imply that lines must be distinct.

For example

- "Let L & M be lines" is vague. It could refer to lines L, M as in any of our examples #4, 5, 6 above. (or even other configurations)

- "Let L & M be <u>distinct lines</u>" is clearer. It means that we are ruling out configuration #6 above.

Meeting Part 3: Parallel Lines

Here are two more definitions

Definition

Words: Lines L & M intersect

Meaning: There is at least one point P that lies on both lines

Definition

Symbol: $L \parallel M$

Spoken: L is parallel to M

Meaning: L & M do not intersect. That is there is no point that lies on both lines.

Three Parallel Postulates

In book Section 2.3, on page 21, three statements about parallel lines are presented. They are called "Parallel Postulates"

Euclidean Parallel Postulate:

For every line L and for every point P that does not lie on L, there is exactly one line M such that P lies on M and M is parallel to L.

Hyperbolic Parallel Postulate:

For every line L and for every point P that does not lie on L, there are no lines M such that P lies on M and M is parallel to L.

Hyperbolic Parallel Postulate:

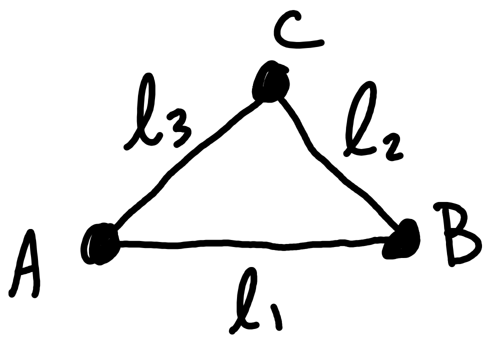
For every line L and for every point P that does not lie on L, there are at least two distinct lines M + N such that P lies on both M + N and both M and N are parallel to L.

Realize that the three Parallel Postulates are not necessarily true or false. In certain situations, one or another of them might be true, or it might be false.

We will encounter the three parallel postulates throughout our course.

For now, we'll consider three examples of Incidence Geometries and discuss the truth of the various parallel postulates

[Example#7]



Notice that in this Example of Incidence Geometry, there are <u>no parallel lines</u>. We can conclude that

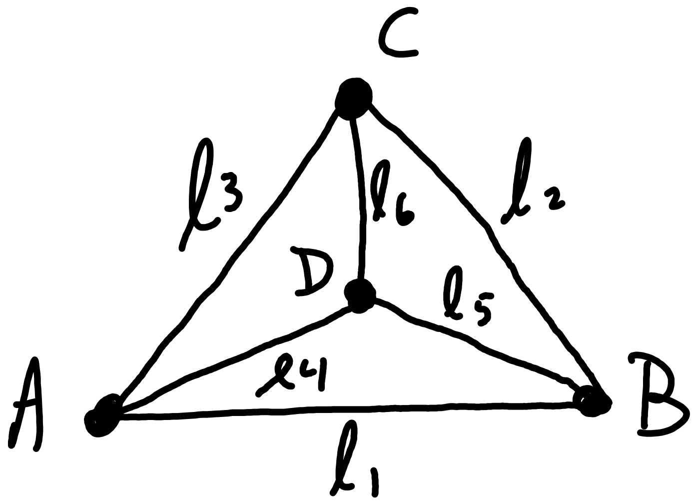
The Euclidean Parallel Postulate is False

The Elliptic Parallel Postulate is <u>True</u>

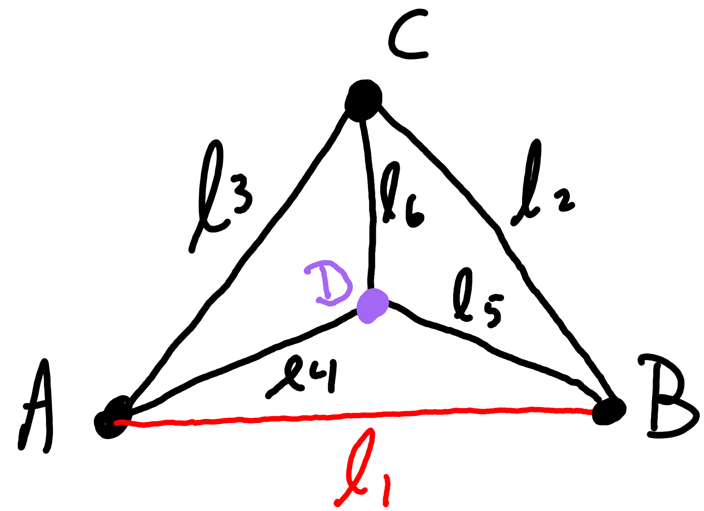
The Hyperbolic Parallel Postulate is <u>False</u>.

End of [Example #7]

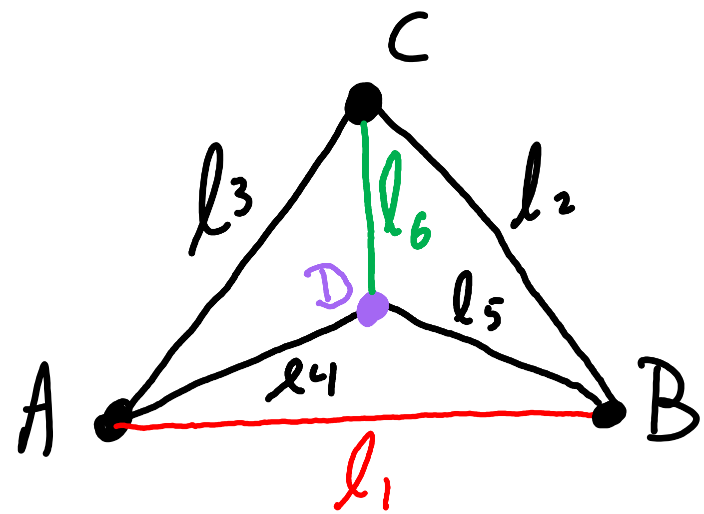
[Example #8]



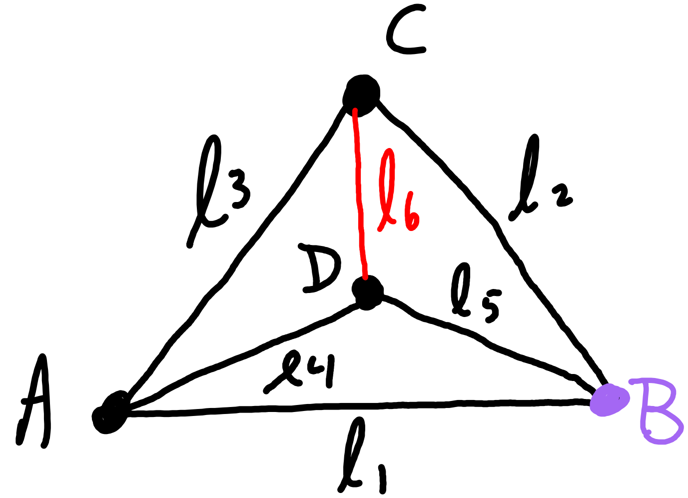
Consider point D that does <u>not</u> lie on line l_1



We see that only line l_6 has the property that D lies on l_6 and $l_6 \parallel l_1$

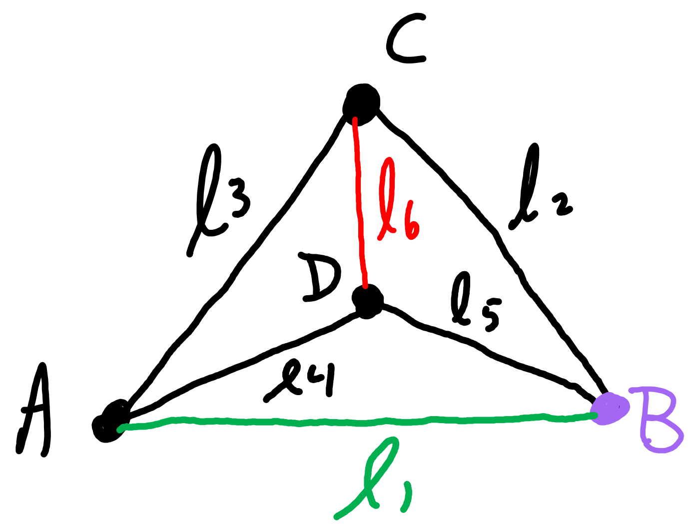


Consider point B that does not lie on line ℓ_6.



We see that only line ℓ_1 has the property that B lies on ℓ_1

and $\ell_1 \parallel \ell_6$.



You can consider some more cases, if you want. (It is actually possible to consider all the possible cases.) But I think you will probably agree that the truth of the three postulates is as follows for [Example #8]

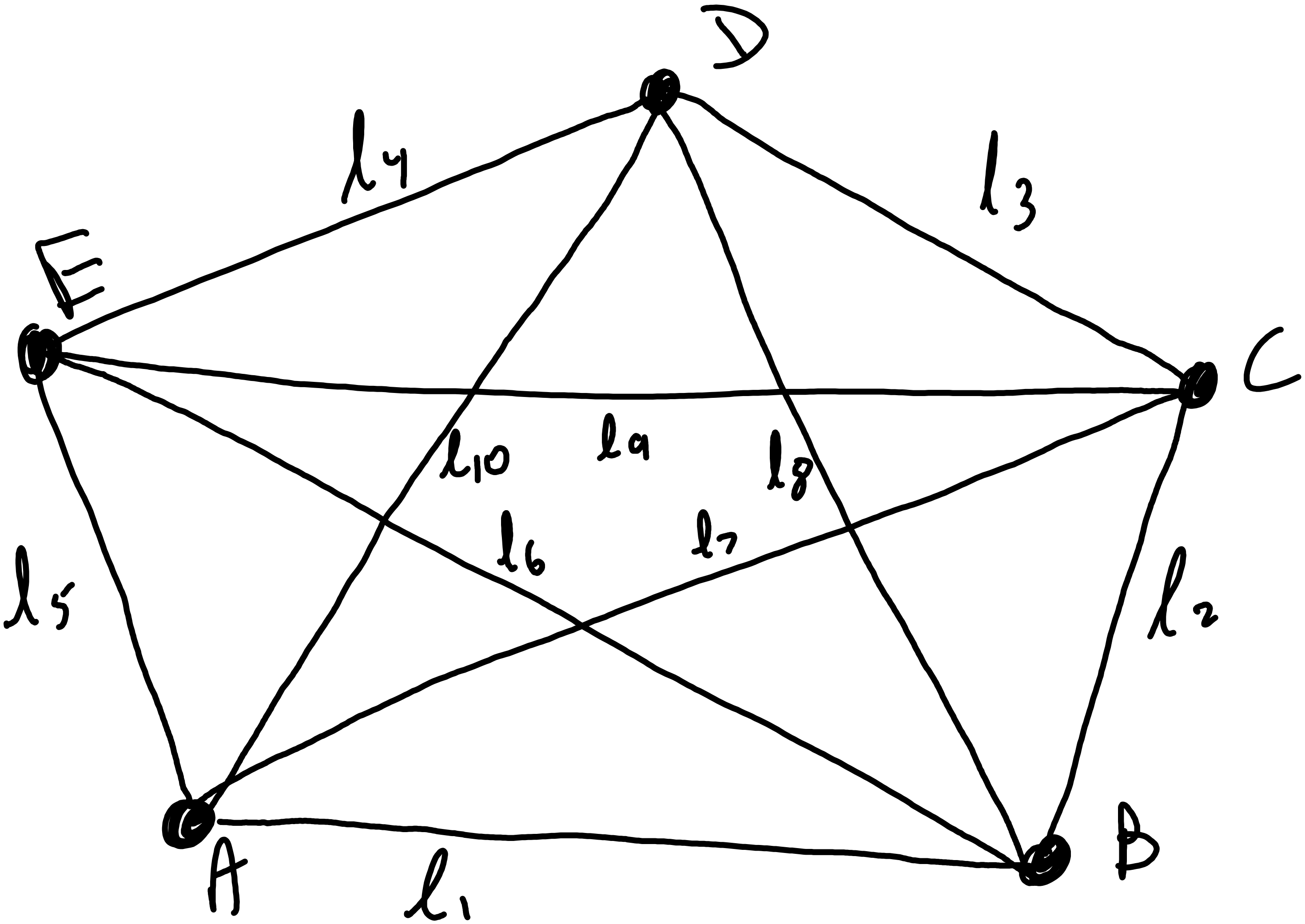
Euclidean: True

Elliptic: False

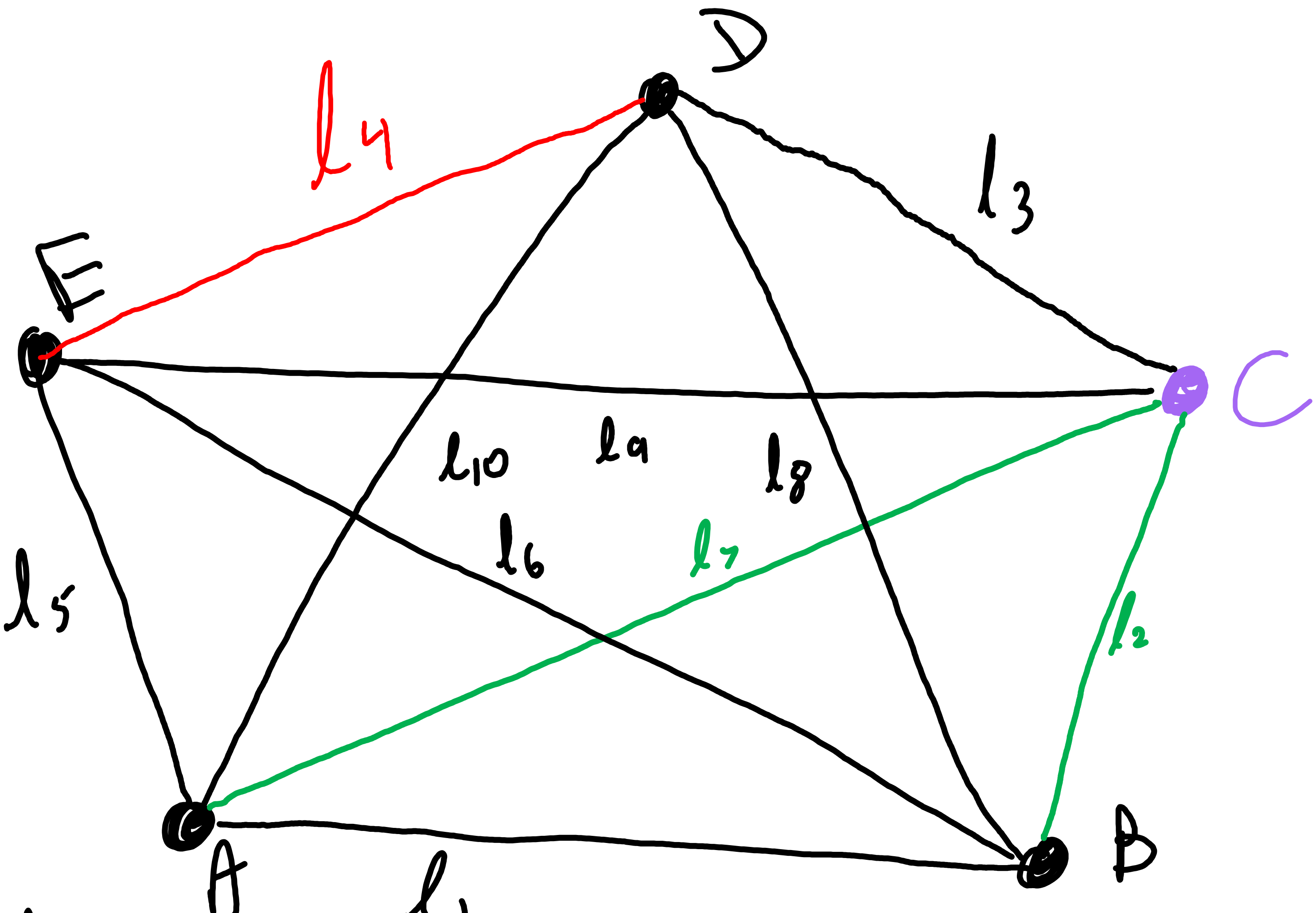
Hyperbolic: False

End of [Example #8]

[Example #9] Consider this example of an Incidence Geometry that has five points and ten lines.

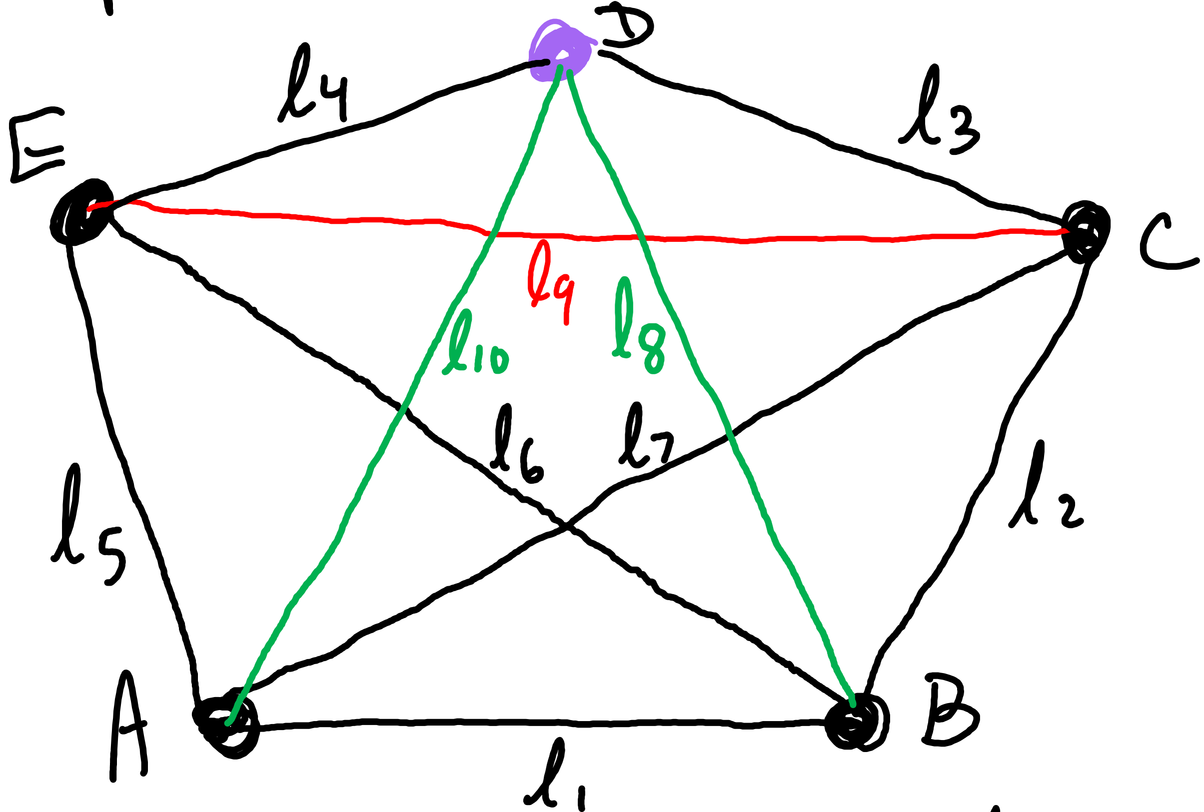


Consider point C that does not lie on line l_4



Only lines l_2, l_7 have the property that C lies on the line and the line is parallel to line l_4

Now consider point D that does not lie on line l_9.



Only lines l_8, l_{10} have the property that point D lies on the line and line $\parallel l_9$.

(Realize that although drawn lines l_9 and l_{10} cross in the drawing, but there is no point at the location where the drawn lines cross. The only points are A, B, C, D, E. So it is true that $l_9 \parallel l_{10}$.)

You can consider some more cases, if you want. (It is actually possible to consider all the possible cases.) But I think you will probably agree that the truth of the three postulates is as follows for [Example #3]

Euclidean: False

Elliptic: False

Hyperbolic: True

[End of Example #9]

End of Lecture