Handout for MATH 1300 (Barsamian) Lecture L22 Thu Oct 8, 2026 Section 3.1

Definition of One-to-One Functions
Words: function f is ***one-to-one***.
Alternate Words: function f is **injective**.
Usage: There is a function f with domain D.
Meaning in symbols: For every $a, b \in D$, if $f(a) = f(b)$ then $a = b$
Meaning in words: If two inputs produce the same output, then they must have been the same input.
Contrapositive Meaning in symbols: For every $a, b \in D$, if $a \neq b$ then $f(a) \neq f(b)$
Meaning in words: If two inputs are not the same, then they produce outputs that are not the same.
Graphical Significance: No horizontal line touches the graph at more than one at x value in the domain D. If some horizontal line does touch the graph at more than one at x value in the domain D, then the function is not one-to-one. (This is a ***horizontal line test***. Note that it is not the only horizontal line test. There is another horizontal line test, used to test for something else.)

You have a homework problem about scrutinizing graphs of functions and deciding whether they are one-to-one. The idea of that problem is tu use the ***horizontal line test*** described above.

Definition of Inverse Function
Words: g is the inverse function of f.
Usage: There is a function f with domain A and range B, and f is one-to-one on domain A.
And there is a function g with domain B and range A.
Meaning: The following two conditions are both satisfied. These conditions are called the ***cancellation equations***.

- For every $a \in A$ (that is, for every a in set A), $(g \circ f)(a) = a$
- For every $b \in B$ (that is, for every b in set B), $(f \circ g)(b) = b$

Fact: If such an inverse function g exists, it can be proved that g is one-to-one on its domain B.
Additional notation: The inverse function g is also denoted f^{-1}. So f^{-1} is a function with domain B and range A such that The following two conditions are both satisfied.

- For every $a \in A$ (that is, for every a in set A), $(f^{-1} \circ f)(a) = a$
- For every $b \in B$ (that is, for every b in set B), $(f \circ f^{-1})(b) = b$

Graphical Significance: Point (a, b) is on the graph of $f \Leftrightarrow$ point (b, a) is on the graph of f^{-1}.
That is, the points on the graph of f^{-1} can be obtained by interchanging the x, y values of all points on the graph of f. Put another way, the graph of f^{-1} can be obtained by reflecting the graph of f across the line $y = x$.

Book's Procedure to Find the Equation of the Inverse Function for a Given Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows.

Step 1: Replace $f(x)$ by y

Step 2: Interchange x and y

Step 3: Solve for y

Step 4: Replace y by $f^{-1}(x)$.

My Procedure to Find the Equation of the Inverse Function for a Given Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows.

Step 1: Examine the equation $y = f(x)$. It is an equation involving x and y, and it is solved for y in terms of x. Make note of the domain A of f. The domain A is the set of inputs that can be fed into f that will produce outputs. Make note of the range B of f. The range B is the set of outputs from f.

Step 2: Solve the equation for x in terms of y. The resulting equation gives $x = f^{-1}(y)$. Note that the domain of f^{-1} is the set B; the range of f^{-1} is the set A.

Step 3: If you prefer to have the variables in your functions be x for the input and y for the output, then you can interchange x and y now. Or you can use some other variable for the input, or even write f^{-1} as an ***empty function***, $f^{-1}(\ )$.

[1] Determine whether the two functions are inverses by checking to see if they satisfy the ***cancellation equations***. **(a)** $f(x) = \frac{x-2}{x-3}$ and $g(x) = \frac{x-3}{x-2}$. **(b)** $f(x) = \frac{x-2}{x-3}$ and $h(x) = \frac{3x-2}{x-1}$

[2] Consider the function $f(x) = \frac{2x+5}{3x-4}$.

(a) What is the domain of f? What is the range of f? (The range question may be hard to answer.)

(b) Find the inverse function, $f^{-1}(x)$. What is the domain of f^{-1}? What is the range of f^{-1}?

(c) Find $\big(f(x)\big)^{-1}$. What is the domain of $\big(f(x)\big)^{-1}$? What is the range of $\big(f(x)\big)^{-1}$? ([1], [2] should help remind you that the *inverse function*, $f^{-1}(x)$ is not the same thing as $\big(f(x)\big)^{-1}$.)

[3] Consider the function $f(x) = \sqrt{x-1} - 2$.

(a) What is the natural domain A for the function f? What is the natural range, B?

(b) Graph f. From your graph of f, you can see that f is one-to-one on its natural domain, because it passes the horizontal line test. So the function $f(x)$ will have an inverse function denoted f^{-1}. What will be the domain and range for the inverse function?

(c) Find the formula for the inverse function.

[4] Consider the function $f(x) = (x-3)^2 + 5$.

(a) Graph f. From your graph of f, you can see that f is not one-to-one on its natural domain, because it fails the horizontal line test. So the function $f(x)$ does not have an inverse function.

(b) What is the domain of f? What is the range?

(c) What is a smaller domain A on which f *would* be one-to-one? Describe A using interval notation. If f is restricted to that smaller domain A, what is the range, B?

(d) The restricted function f is one-to-one, it does have an inverse function, which we'll denote as f^{-1}. Find the formula for f^{-1}. What is the domain of f^{-1}? What is the range?