Handout: Implicit Differentiation (Stewart Section 2.6)

Part 1: Equations – vs – Functions

[Example 1]

- $x^3 + y^3 = 7$ equation involving x and y describes y *implicitly*
- $y = (7 - x^3)^{1/3}$ equation involving x and y, solved for y in terms of x. Gives y as a function of x. Describes y *explicitly*.

Observe that the two equations above express the same relationship between x and y.

[Example 2]

- $x^2 + y^2 = 7$ equation involving x and y describes y *implicitly*. Expresses a relationship between x and y. Cannot be solved for y as a function of x. We know this because the graph is a circle. Fails the vertical line test. Can't be the graph of a function.

Part 2: Implicit Differentiation

Suppose we have equation involving x and y and we want to find $\frac{dy}{dx}$.

If the equation can be solved for y in terms of x. We should do that first

$$y = \text{some expression involving } x$$

then take the ordinary derivative

$$\frac{dy}{dx} = \frac{d}{dx}(\text{some expression involving } x)$$

If the equation *cannot* be solved for y in terms of x, we can still find $\frac{dy}{dx}$ using a method called ***Implicit Differentiation***.

The Method of Implicit Differentiation

(Used for finding y' when x, y are related by an equation that is not solved for y.)

Starting with: An equation involving x and y.

Step 1: Take derivative of left and right sides of this new equation with respect to x. Keep in mind the difference between taking the derivative of x and taking the derivative of y.

$$\frac{d}{dx}x \underset{\substack{power\\rule\\with\ n=1}}{=} 1$$

$$\frac{d}{dx}y = y' \quad \text{This is unknown! We cannot go any farther.}$$

That is, we have to treat the symbol y as if it is an (unknown) function of the variable x. The result will be a new equation involving x and y and y'.

Step 2: Solve for y'. The result will be a new equation of the form

$$y' = \text{expression involving } x \text{ and } y$$

Handout: Related Rates (Stewart Section 2.7)

Recall the Method of Implicit Differentiation

(Used for finding y' when x, y are related by an equation that is not solved for y.)
Starting with: An equation involving x and y.
Step 1: Take derivative of left and right sides of this new equation with respect to x. Keep in mind the difference between taking the derivative of x and taking the derivative of y.

$$\frac{d}{dx}x \underset{\substack{power\\rule\\with\ n=1}}{=} 1$$

$$\frac{d}{dx}y = y'$$ This is unknown! We cannot go any farther.

That is, we have to treat the symbol y as if it is an (unknown) function of the variable x.
The result will be a new equation involving x and y and y'.
Step 2: Solve for y'. The result will be a new equation of the form

$$y' = \text{expression involving } x \text{ and } y$$

Recall the Definition of Instantaneous of Change for a function $f(t)$

Words: *Instantaneous Rate of Change of* $f(t)$ *at time* $t = a$.
Symbol: $f'(a)$
Spoken: The derivative of f at a.
Units: The units of $f'(a)$ are $\frac{\text{units of the quantity } f(t)}{\text{units of time } t}$
Additional terminology: When the variable is t, representing time and the function $f(t)$ is a position function, representing the position of an object at time t, then the Instantaneous rate of change is called the *instantaneous velocity at time* $t = a$.

Related Rates problems:

Given various quantities that are related by an equation,
and given values for certain of the quantities or rates of change of those quantities
Find some unknown rate of change, or find some unknown quantity.

Solving Related Rates Problems

1) Draw a picture of the situation
2) Label the picture with variables for important quantities
3) Label the picture with numbers for known quantities (or known rates)
4) Identify the goal: find some unknown rate or find some unknown quantity.
5) Find an equation relating the variables.
6) Differentiate both sides of this equation with respect to time t using the method of ***Implicit Differentiation***. The result will be a **new equation** relating the quantities and the derivatives of the quantities. That is, this new equation describes a relationship between the rates of change of the quantities. Hence, the name ***Related Rates***.
7) Solve this new equation for the unknown derivative or the unknown quantity.
8) Substitute in values.