Limits Video A: Graphical Approach

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Topics in this Video

- Terminology and Notation of Functions
- Definition of Limit
- Examples of function values and limits for a function given by a graph
- One-Sided Limits
- Example involving producing a graph with specified limit behavior

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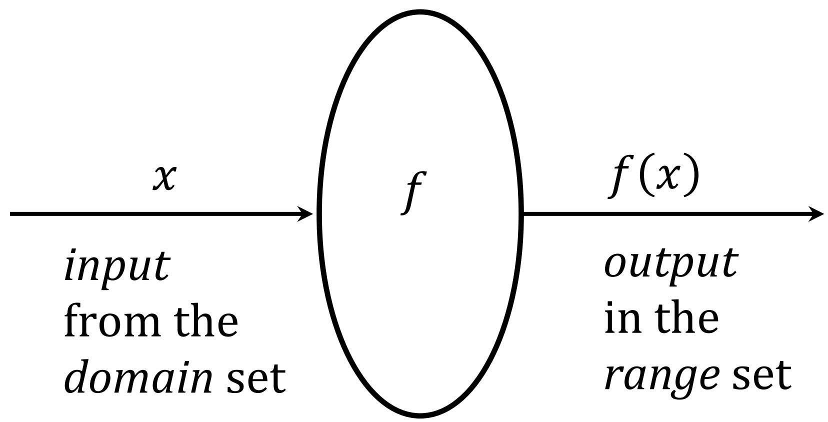
Terminology and Notation of Functions

This video is about *limits*. Before discussing *limits*, it is useful to review the terminology and notation of *functions*.

A *function* $f(x)$ can be thought of as a machine that takes as input a number from a set of real numbers called the *domain*, and produces as output a number in a set of real numbers called the *range*. This can be visualized in the following *machine diagram*.



It is important to note that

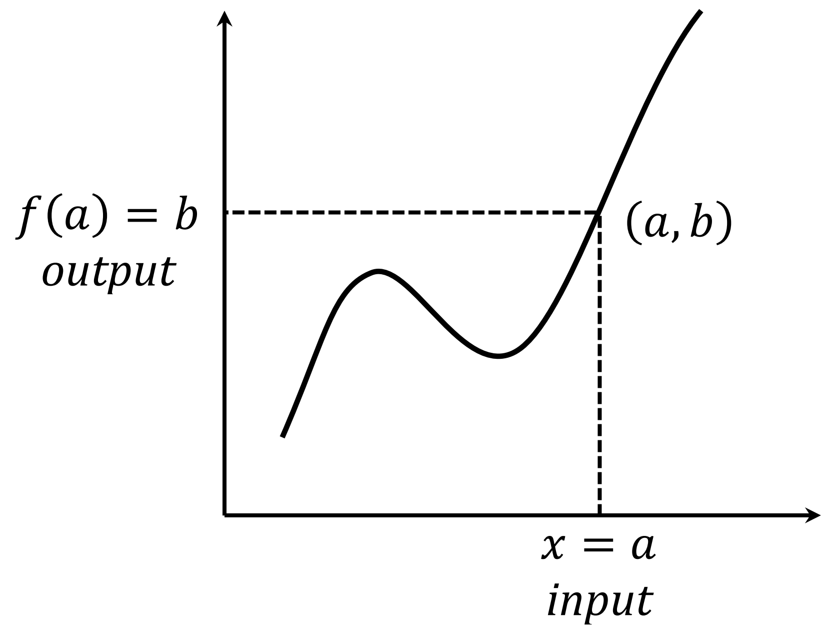
- The symbol f is the *name* of the function.
- The symbol x denotes the *input*.
- The symbol $f(x)$ denotes the resulting *output*.

Displaying Numerical Functions on Graphs

The horizontal axis (the x axis) is used for the input numbers.

The vertical axis (the y axis) is used for the output numbers.

If an *input* $x = a$ causes an *output* $f(a) = b$, then the *point* $(x, y) = (a, b)$ is on the graph, and vice-versa.



So for instance, the symbol $f(2) = 7$ would tell us

- For the function f, an *input* of $x = 2$ causes an *output* of $y = 7$.
- The *point* $(x, y) = (2,7)$ is on the graph of f.

Limits

Now we are ready to discuss *limits*. We start with the definition.

The Definition of *Limit*

Symbol: $\lim_{x \to c} f(x) = L$.

Spoken: "The limit, as x approaches c, of $f(x)$ is L."

Less-Abbreviated Symbol: $f(x) \to L$ as $x \to c$.

Spoken: "$f(x)$ approaches L as x approaches c."

Usage: x is a variable, f is a function, c is a real number, and L is a real number.

Meaning: as x gets closer and closer to c, but not equal to c, the value of $f(x)$ gets closer and closer to L (may actually equal L).

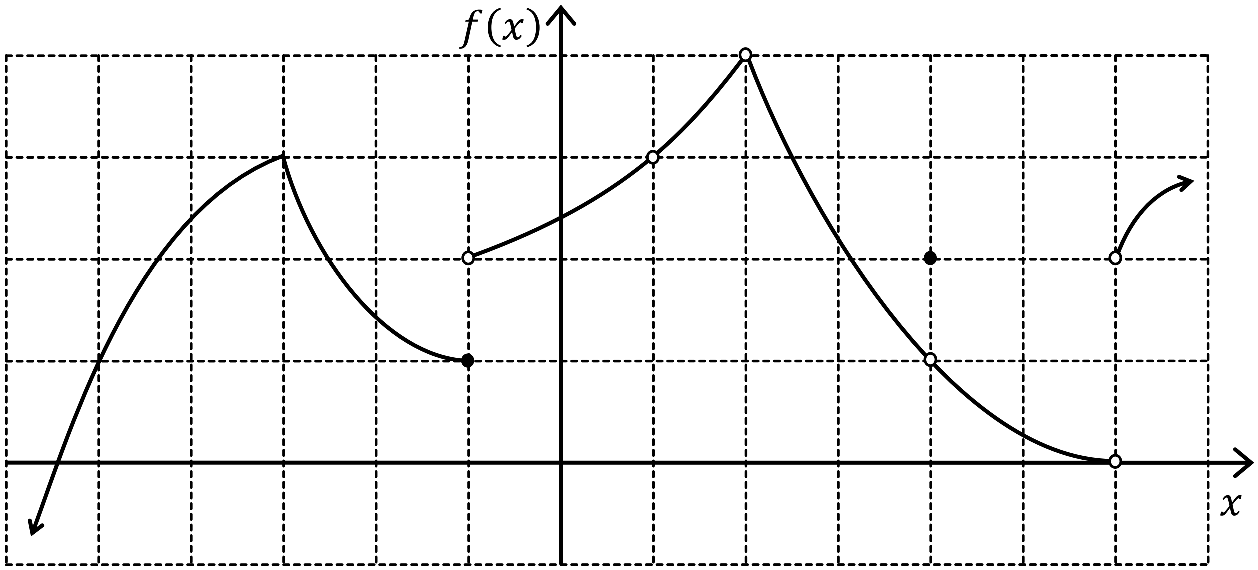
Graphical Significance: We'll come back to this after a couple of examples.

In this video, we explore limits using a *graphical* approach (the function f is given by a *graph*, not by a *formula*.)

We will start by considering examples of the following kind:

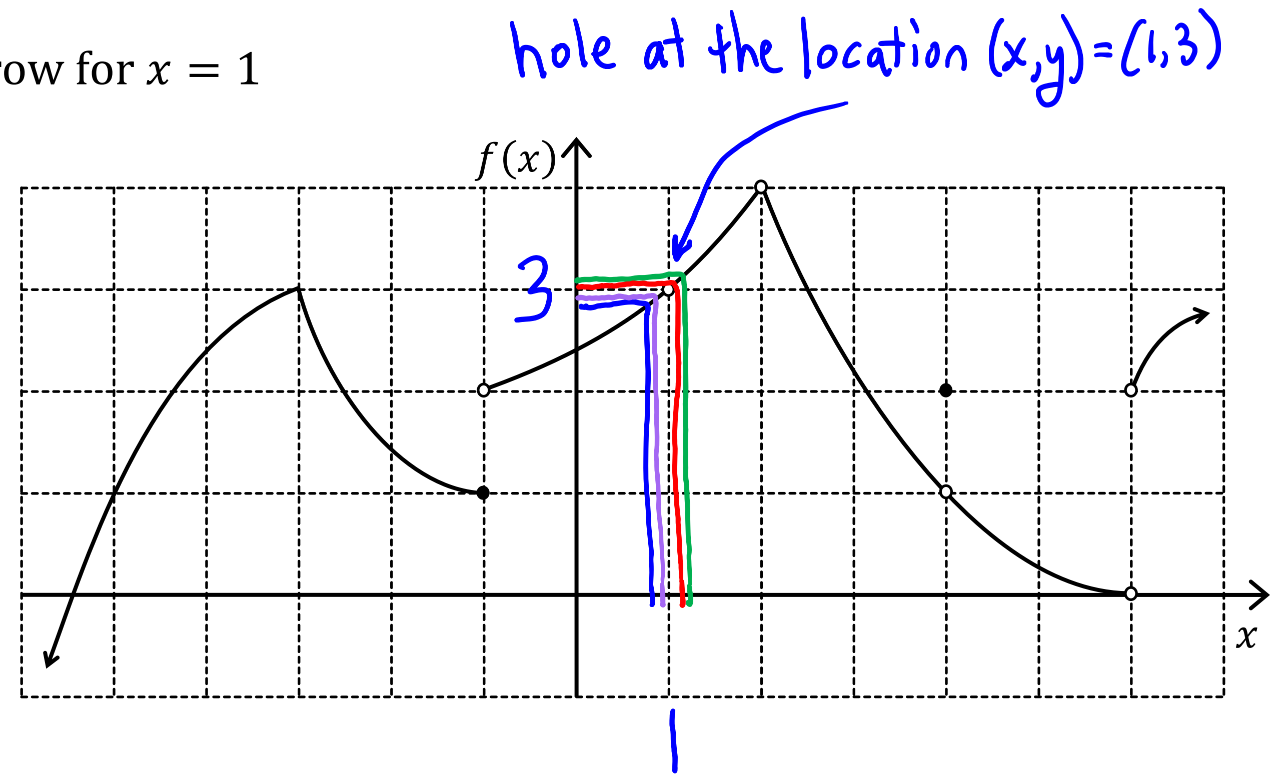
Given graph of f ➔ give a description of limit behavior of f.

Extended [Example 1]: Limits for a Function Given by a Graph



x-value	limit from left	limit from right	limit	y-value
-3	$\lim_{x\to -3^-} f(x)$	$\lim_{x\to -3^+} f(x)$	$\lim_{x\to -3} f(x)$	$f(-3)$
-1	$\lim_{x\to -1^-} f(x)$	$\lim_{x\to -1^+} f(x)$	$\lim_{x\to -1} f(x)$	$f(-1)$
1	$\lim_{x\to 1^-} f(x)$	$\lim_{x\to 1^+} f(x)$	$\lim_{x\to 1} f(x)$	$f(1)$
4	$\lim_{x\to 4^-} f(x)$	$\lim_{x\to 4^+} f(x)$	$\lim_{x\to 4} f(x)$	$f(4)$
6	$\lim_{x\to 6^-} f(x)$	$\lim_{x\to 6^+} f(x)$	$\lim_{x\to 6} f(x)$	$f(6)$

Start with the row for $x = 1$



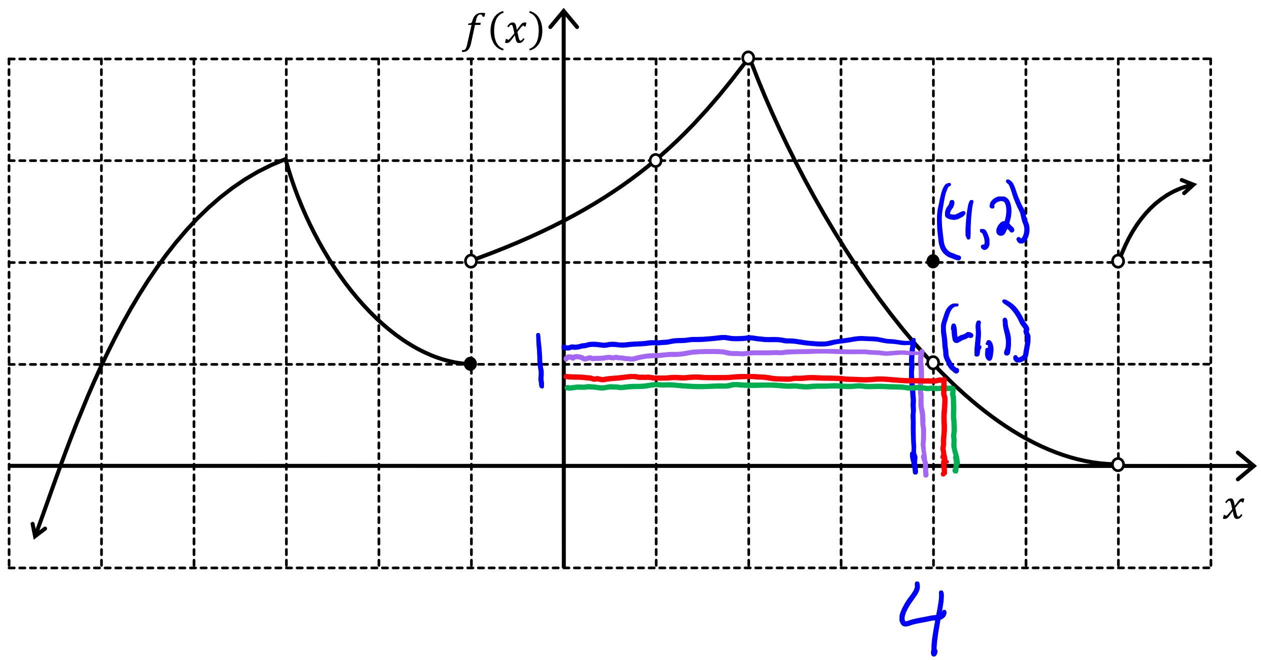
Observations:

- The graph has no y value (there is no point on the graph) at $x = 1$. So $f(1)$ does not exist.
- When x gets closer & closer to 1, but not equal to 1, the y values get closer & closer to 3.
- (We say that there is a *hole* in the graph at the location $(x, y) = (1,3)$.)

Abbreviations of these observations in math symbols:

x-value	limit from left	limit from right	limit	y-value
1	$\lim_{x \to 1^-} f(x)$	$\lim_{x \to 1^+} f(x)$	$\lim_{x \to 1} f(x) = 3$	$f(1)$ DNE

Do the row for $x = 4$



Observations:

- There is a point on the graph at the location $(x, y) = (4,2)$. So $f(4) = 2$.
- When x gets closer & closer to 4, but not equal to 4, the y values get closer & closer to 1. (But there is a hole in the graph at the location $(x, y) = (4,1)$.)

Abbreviations of these observations in math symbols:

x-value	limit from left	limit from right	limit	y-value
4	$\lim_{x \to 4^-} f(x)$	$\lim_{x \to 4^+} f(x)$	$\lim_{x \to 4} f(x) = 1$	$f(4) = 2$

Add a line to the definition of limit:

The Definition of *Limit*

Symbol: $\lim_{x \to c} f(x) = L$.

Spoken: "The limit, as x approaches c, of $f(x)$ is L."

Less-Abbreviated Symbol: $f(x) \to L$ as $x \to c$.

Spoken: "$f(x)$ approaches L as x approaches c."

Usage: x is a variable, f is a function, c is a real number, and L is a real number.

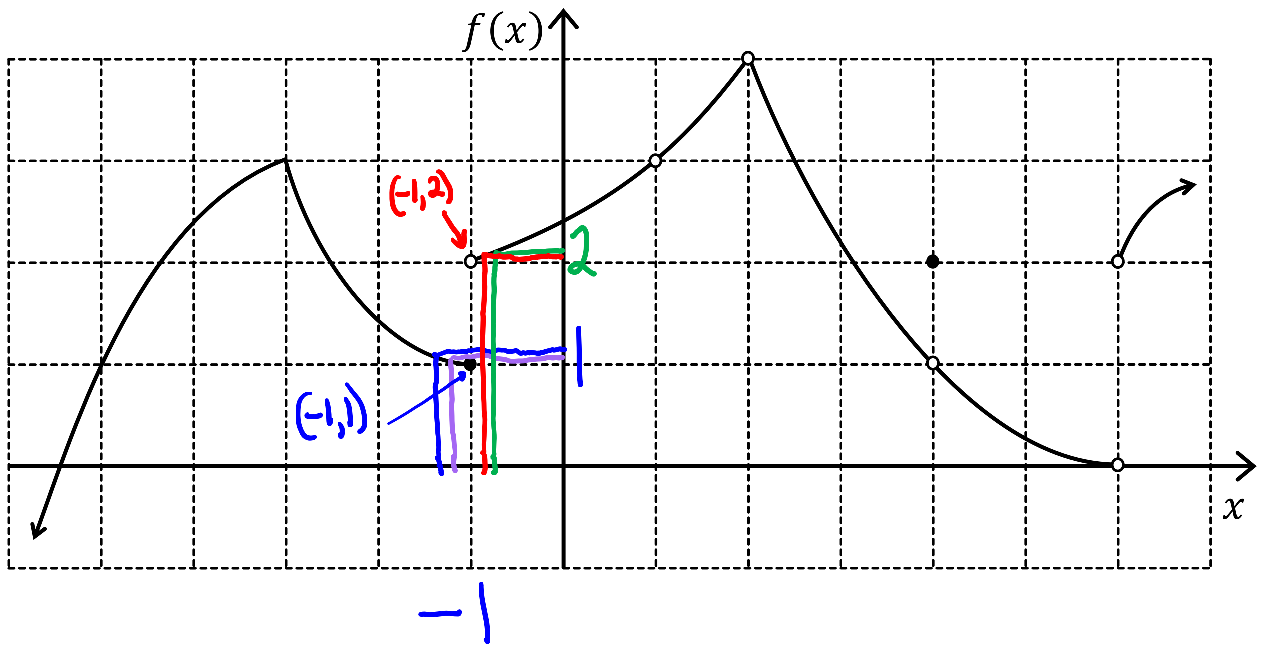
Meaning: as x gets closer and closer to c, but not equal to c, the value of $f(x)$ gets closer and closer to L (may actually equal L).

Graphical Significance: The graph of f appears to be heading for location $(x, y) = (c, L)$ from both sides.

And note the difference between the symbols $f(c)$ and $\lim_{x \to c} f(x)$.

- The symbol $f(c)$ denotes the y value at the x value $x = c$.
- The symbol $\lim_{x \to c} f(x)$ tells us about the *trend* in the y values as x gets closer and closer to c.

Do the row for $x = -1$



Observations:

- There is a point on the graph at the location $(x, y) = (-1,1)$. So $f(-1) = 1$.
- As x gets closer and closer to -1, but not equal to -1, there is no single y value that all of the y values are getting closer and closer to. We could also say that as x gets closer and closer to -1, there is no single (x, y) location that the graph is heading towards.

x-value	limit from left	limit from right	limit	y-value
-1	$\lim_{x\to -1^-} f(x)$	$\lim_{x\to -1^+} f(x)$	$\lim_{x\to -1} f(x)$ DNE	$f(-1) = 1$

As we just observed, the $\lim_{x \to -1} f(x)$ does not exist because there is no single (x, y) location that the graph is heading towards.

But there are some obvious *trends* in the graph:

- On the left side of $x = -1$, the graph appears to be heading for the location $(x, y) = (-1,1)$.
- On the right side of $x = -1$, the graph appears to be heading for the location $(x, y) = (-1,2)$.

It would be useful to have some terminology and notation for those trends. That is the idea of *one-sided limits*. The definitions follow on the next page.

The Definition of *Limit from the Left*

Symbol: $\lim_{x \to c^-} f(x) = L$.

Spoken: "The limit, as x approaches c from the left, of $f(x)$ is L."

Meaning: as x gets closer and closer to c, but less than c, the value of $f(x)$ gets closer and closer to L (may actually equal L).

Graphical Significance: The graph of f appears to be heading for location $(x, y) = (c, L)$ from the left.

The Definition of *Limit from the Right*

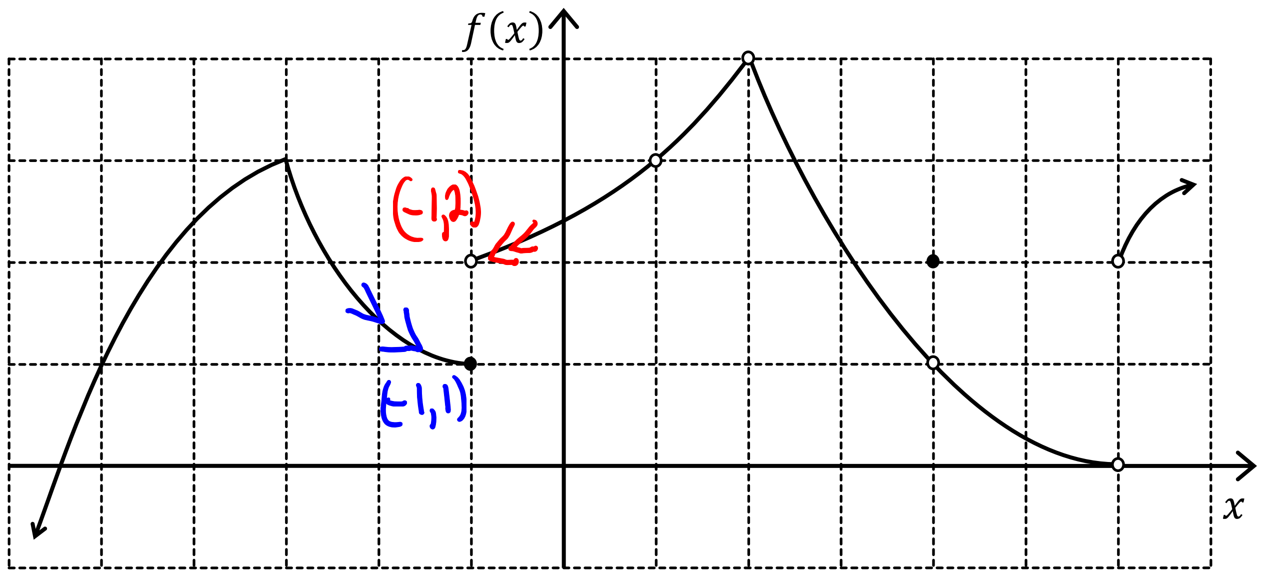
Symbol: $\lim_{x \to c^+} f(x) = L$.

Spoken: "The limit, as x approaches c from the right, of $f(x)$ is L."

Meaning: as x gets closer and closer to c, but greater than c, the value of $f(x)$ gets closer and closer to L (may actually equal L).

Graphical Significance: The graph of f appears to be heading for location $(x, y) = (c, L)$ from the right.

Finish row $x = -1$.



x-value	limit from left	limit from right	limit	y-value
-1	$\lim_{x\to -1^-} f(x) = 1$	$\lim_{x\to -1^+} f(x) = 2$	$\lim_{x\to -1} f(x)\ DNE$	$f(-1) = 1$

Re-cast the definition of *Limit* using 3-part test involving one-sided limits.

The Definition of *Limit* written as a 3-part test involving O*ne-Sided Limits*.

Symbol: $\lim_{x \to c} f(x) = L$.

Meaning: The function passes this three-part test

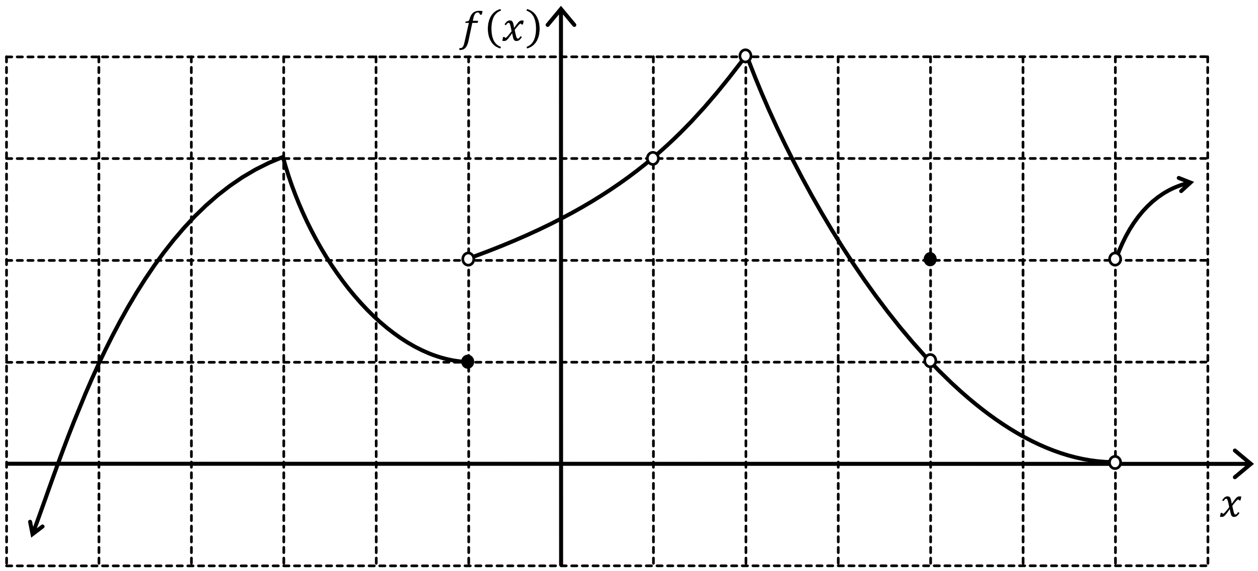
test a: The limit from the left, $\lim_{x \to c^-} f(x)$, exists

test b: The limit from the right, $\lim_{x \to c^+} f(x)$, exists

test c: The values of the limits from the left and right match, with value L. That is,

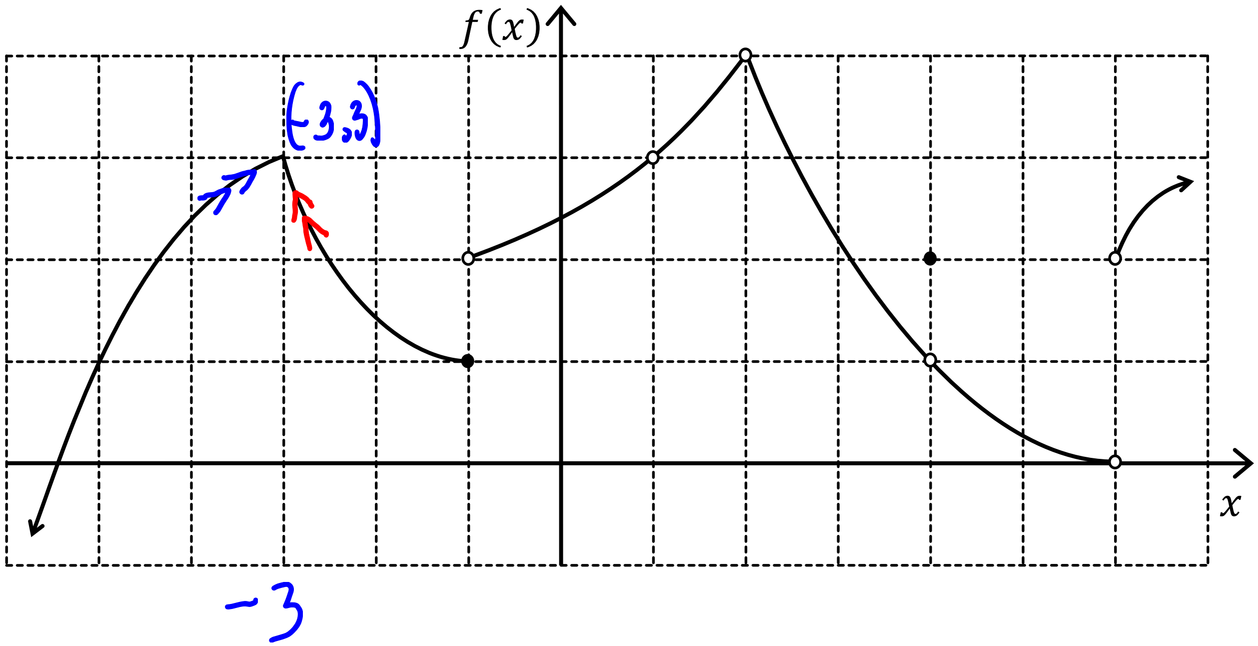
$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x).$$

Go back and fill in empty cells in rows $x = 1$ and $x = 4$.



x-value	limit from left	limit from right	limit	y-value
1	$\lim_{x\to 1^-} f(x) = 3$	$\lim_{x\to 1^+} f(x) = 3$	$\lim_{x\to 1} f(x) = 3$	$f(1)\ DNE$
4	$\lim_{x\to 4^-} f(x) = 1$	$\lim_{x\to 4^+} f(x) = 1$	$\lim_{x\to 4} f(x) = 1$	$f(4) = 2$

Do the row for $x = -3$.

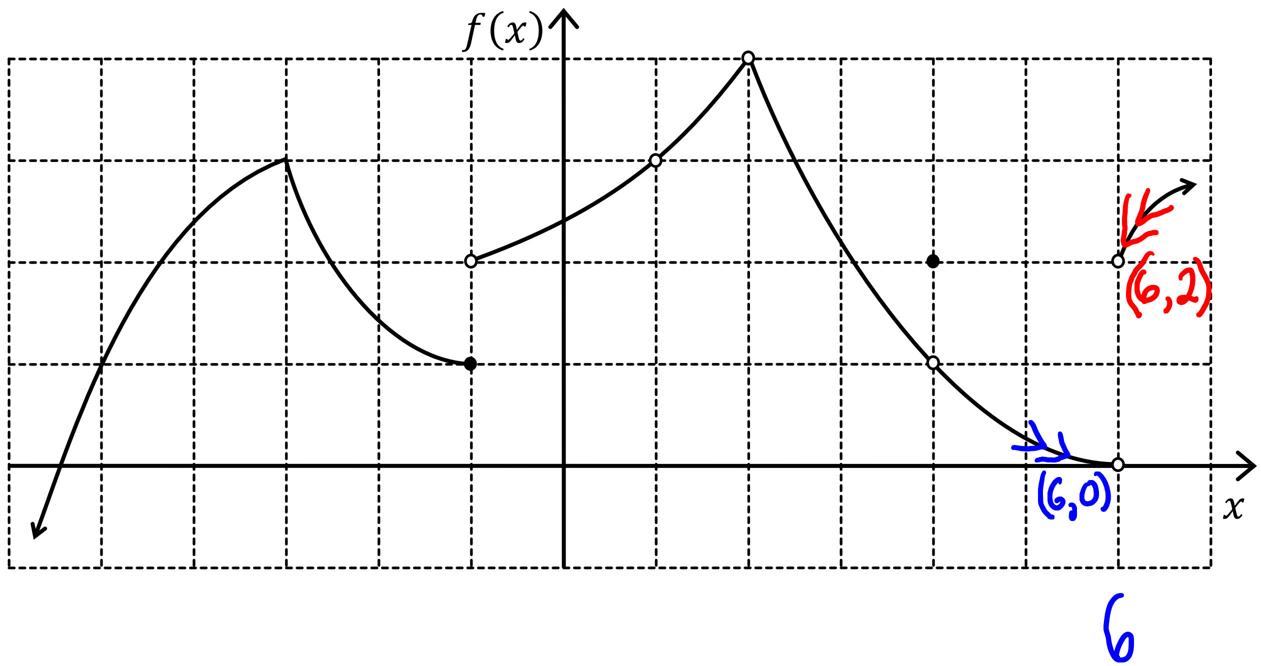


Observations:

- There is a point on the graph at the location $(x, y) = (-3,3)$.
- When x gets closer and closer to -3, but not equal to -3, the y values get closer and closer to 3. That is, the graph appears to be heading for the location $(x, y) = (-3,3)$.

x-value	limit from left	limit from right	limit	y-value
-3	$\lim_{x \to -3^-} f(x) = 3$	$\lim_{x \to -3^+} f(x) = 3$	$\lim_{x \to -3} f(x) = 3$	$f(-3) = 3$

Finally, do the row for $x = 6$

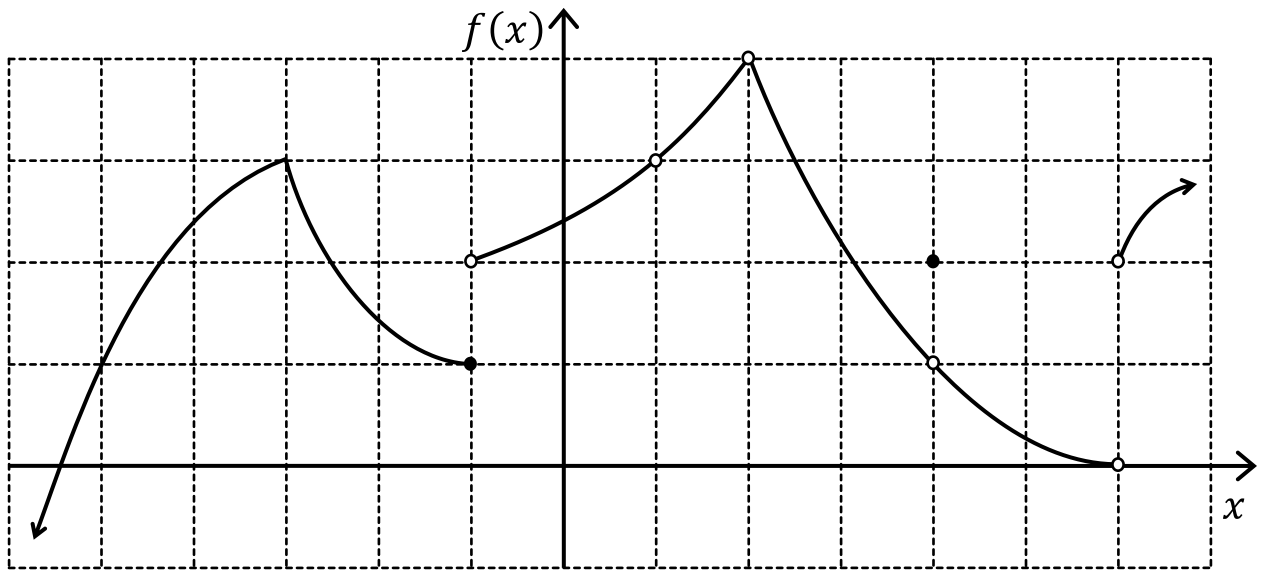


Observations:

- There is no point on the graph with $x = 6$.
- On the left side of $x = 6$, the graph appears to be heading for the location $(x, y) = (6,0)$.
- On the right side of $x = 6$, the graph appears to be heading for the location $(x, y) = (6,2)$.

x-value	limit from left	limit from right	limit	y-value
6	$\lim_{x\to 6^-} f(x) = 0$	$\lim_{x\to 6^+} f(x) = 2$	$\lim_{x\to 6} f(x)$ DNE	$f(6)$ DNE

Gathering up all of our results in one table:



x-value	limit from left	limit from right	limit	y-value
-3	$\lim_{x\to -3^-} f(x) = 3$	$\lim_{x\to -3^+} f(x) = 3$	$\lim_{x\to -3} f(x) = 3$	$f(-3) = 3$
-1	$\lim_{x\to -1^-} f(x) = 1$	$\lim_{x\to -1^+} f(x) = 2$	$\lim_{x\to -1} f(x)\ DNE$	$f(-1) = 1$
1	$\lim_{x\to 1^-} f(x) = 3$	$\lim_{x\to 1^+} f(x) = 3$	$\lim_{x\to 1} f(x) = 3$	$f(1)\ DNE$
4	$\lim_{x\to 4^-} f(x) = 1$	$\lim_{x\to 4^+} f(x) = 1$	$\lim_{x\to 4} f(x) = 1$	$f(4) = 2$
6	$\lim_{x\to 6^-} f(x) = 0$	$\lim_{x\to 6^+} f(x) = 2$	$\lim_{x\to 6} f(x)\ DNE$	$f(6)\ DNE$

End of Extended [Example 1]

Example of a different type:

Given a description of limit behavior of f ➔ sketch a possible graph of f

[Example 2] Sketch a graph that satisfies all these conditions:

$$f(1) = 3$$

$$\lim_{x \to 1^-} f(x) = 2$$

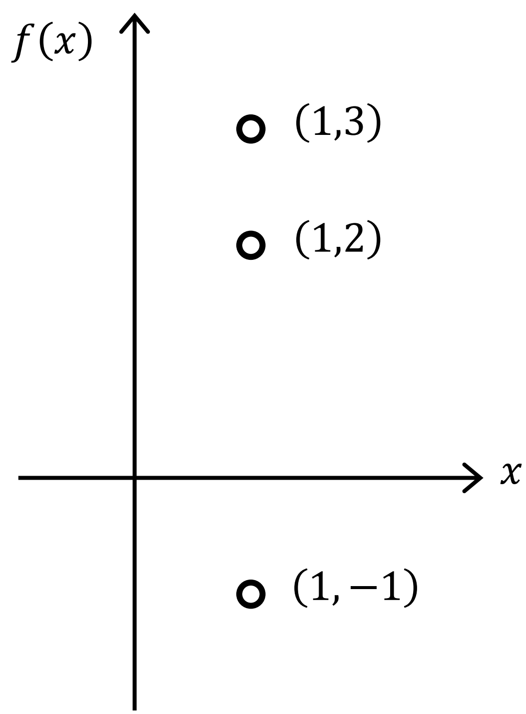
$$\lim_{x \to 1^+} f(x) = -1$$

Solution:

Start by noting that in the given information, three (x, y) locations are implicated.

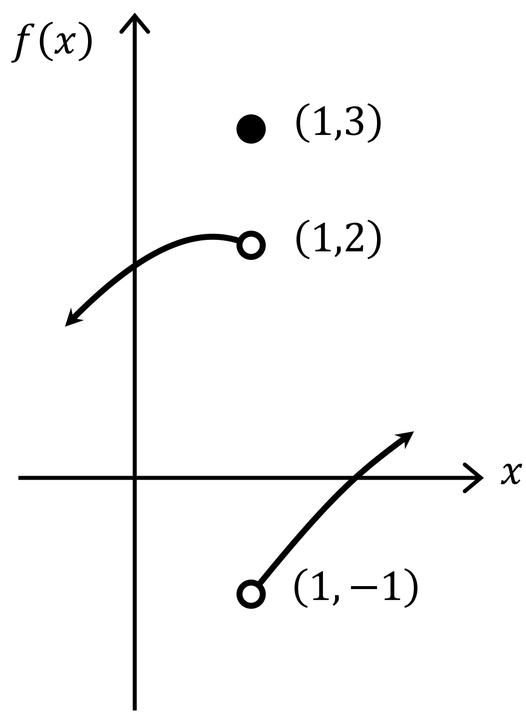
- The symbol $f(1) = 3$ is about the location $(x, y) = (1,3)$.
- The symbol $\lim_{x\to 1^-} f(x) = 2$ is about the location $(x, y) = (1,2)$.
- The symbol $\lim_{x\to 1^+} f(x) = -1$ is about the location $(x, y) = (1, -1)$.

On one set of axes, plot these three locations with open circles and label the locations with their (x, y) coodinates.



Then add features that convey what the given information tells us about those locations.

- The symbol $f(1) = 3$ tells us that there is a point on the graph at the location $(x, y) = (1,3)$, so we fill in the open circle at that location.
- The symbol $\lim_{x \to 1^-} f(x) = 2$ tells us that the graph is heading for the location $(x, y) = (1,2)$ from the left, so we draw some sort of smooth curve heading for that location from the left.
- The symbol $\lim_{x \to 1^+} f(x) = -1$ tells us that the graph is heading for the location $(x, y) = (1, -1)$ from the right so we draw a smooth curve heading for that location from the right.



Question: Can we fill in all of the open circles?

Answer: No!

If we filled in more than one circle, then that would mean that for $x = 1$, there is more than one y value. This would violate the definition of *function*, which says that for a particular *input* (a particular x value), there is exactly one *output* (one y value). (Put another way, the graph would fail the ***vertical line test***.)

End of [Example 2]

End of Video