Limits Video B: Analytical Approach to Limits

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Topics in this Video

- First Example: Estimating the Value of a Limit
- Tools: Theorems Presenting Properties of Limits
- Basic Examples of Computing Limits Analytically

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Useful Definition from the Previous Video

The Definition of *Limit*

Symbol: $\lim_{x \to c} f(x) = L$.

Spoken: "The limit, as x approaches c, of $f(x)$ is L."

Less-Abbreviated Symbol: $f(x) \to L$ as $x \to c$.

Spoken: "$f(x)$ approaches L as x approaches c."

Usage: x is a variable, f is a function, c is a real number, and L is a real number.

Meaning: as x gets closer and closer to c, but not equal to c, the value of $f(x)$ gets closer and closer to L (may actually equal L).

Graphical Significance: The graph of f appears to be heading for location $(x, y) = (c, L)$ from both sides.

And note the difference between the symbols $f(c)$ and $\lim_{x \to c} f(x)$.

- The symbol $f(c)$ denotes the y value at the x value $x = c$.
- The symbol $\lim_{x \to c} f(x)$ tells us about the *trend* in the y values as x gets closer and closer to c.

Topic for this video: We will take an *analytical* approach to limits.

That means that the function f is given by a *formula*, not *graph*.

[Example 1] Estimating the Value of a limit

Let $f(x) = -7x^2 + 13x - 25$.

(A) Find $f(-2)$.

Solution:

$$f(-2) = -7(-2)^2 + 13(-2) - 25 = -7(4) - 26 - 25 = -79.$$

(B) Use a table of x, y values to *estimate* the value of $\lim_{x \to -2} f(x)$.

Solution: The symbol is telling us that we need to consider what happens to the values of $f(x)$ when x gets closer and closer to -2 but not equal to -2.

We can experiment by making a table of x and $f(x)$ values. Notice in the left column, we put values of x that are getting closer and closer to -2 but not equal to -2. In the right column, we put the resulting values of $f(x)$, found using a calculator.

x	$f(x) = -7x^2 + 13x - 25$
-2.1	$f(-2.1) = -7(-2.1)^2 + 13(-2.1) - 25 \underset{\text{calculator}}{=} -83.1700$
-2.01	$f(-2.01) = -7(-2.01)^2 + 13(-2.01) - 25 \underset{\text{calculator}}{=} -79.4107$
-2.001	$f(-2.001) = -7(-2.001)^2 + 13(-2.001) - 25 \underset{\text{calculator}}{=} -79.0410$
-2.0001	$f(-2.0001) = -7(-2.0001)^2 + 13(-2.0001) - 25 \underset{\text{calculator}}{=} -79.0041$

It looks like the values of $f(x)$ are getting closer and closer to -79.

The table that we just built had values of x that are getting closer and closer to -2 and not equal to -2, but they also had the property that the x values were all *less than* -2. We should also build a table with values of x that are getting closer and closer to -2 but always *greater than* -2.

x	$f(x) = -7x^2 + 13x - 25$
-1.9	$f(-1.9) = -7(-1.9)^2 + 13(-1.9) - 25 \underset{\text{calculator}}{=} -74.9700$
-1.99	$f(-1.99) = -7(-1.99)^2 + 13(-1.99) - 25 \underset{\text{calculator}}{=} -78.5907$
-199.9	$f(-1.999) = -7(-1.999)^2 + 13(-1.999) - 25 \underset{\text{calculator}}{=} -78.95907$
-1.9999	$f(-1.9999) = -7(-1.9999)^2 + 13(-1.999) - 25 \underset{\text{calculator}}{=} -78.9959$

In this table, it looks like the values of $f(x)$ are getting closer and closer to -79.

Based on these two tables, we could write the following observation:

When x gets closer and closer to -2 but not equal to -2, we would *estimate* that the value of $f(x)$ gets closer and closer to -79.

We could abbreviate the above observation using limit notation:

We would *estimate* that $\lim_{x \to -2} f(x) \underset{\text{estimate}}{=} -79$.

End of [Example 1]

Remark #1 About the Result of [Example 1]: Comparison of the limit and the y value

Observe that our estimate of the value of the limit matches the value obtained in our earlier computation of the y value.

Result of (A): $f(-2) = -79$.

Result of (B): $\lim_{x \to -2} f(x) \underset{\text{estimate}}{=} -79$

A natural question is

Question: Does the value of $\lim_{x \to c} f(x)$ always match the value of $f(c)$?

Answer: Remember that in Limits Video A, we saw examples of a function $f(x)$ given by a graph where the value of the limit did not match the y value. In this first example of a function $f(x)$ given by a formula, it happens that the value of the limit does match the y value. But we should not expect that this will aways happen.

Remark #2 About the Result of [Example 1]: Unsatisfying Method

This process used in part (B) should seem very unsatisfying.

- We had to use a *calculator* to find the values of $f(x)$ to fill in two large tables.
- We could only *estimate* that the values of $f(x)$ are getting closer and closer to -79, so we could only *estimate* that the value of the limit is $\lim_{x \to -2} f(x) = -79$.

Question: Is there is a better way? That is, is there some way to analyze the formula for $f(x)$ to determine the value of the limit *precisely*, without *estimating*?

Answer: There *are* analytical techniques, developed in higher-level math, that provide a way of analyzing the formulas for certain kinds of functions to determine their limits.

The analytical techniques, themselves, are beyond the level of an introductory Calculus course. But the general *results* of using the techniques can be presented as *Theorems* that can be used in our course. Three such *Theorems about Limits* are presented on the next two pages.

Three Theorems About Limits.

THEOREM 2 Properties of Limits

Let f and g be two functions, and assume that

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M$$

where L and M are real numbers (both limits exist). Then

1. $\lim_{x \to c} k = k$ for any constant k
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$
4. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$
5. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) = kL$ for any constant k
6. $\lim_{x \to c} [f(x) \cdot g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = LM$
7. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
8. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$ if $L > 0$ or n is odd

THEOREM 3 Limits of Polynomial and Rational Functions

1. $\lim_{x \to c} f(x) = f(c)$ for f any polynomial function.
2. $\lim_{x \to c} r(x) = r(c)$ for r any rational function with a nonzero denominator at $x = c$.

THEOREM 4 Limit of a Quotient

If $\lim_{x \to c} f(x) = L$, $L \neq 0$, and $\lim_{x \to c} g(x) = 0$,

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \qquad \text{does not exist}$$

In this video we will do three examples that use Theorems 2, 3, and 4.

Our first example will be a revisiting of **[Example 1]**. This time, instead of *guessing* the value of the limit, we will find it *precisely*, using a *Theorem*.

[Example 1, continued]

(C) Use the *Theorems About Limits* to find the value of $\lim_{x \to -2} f(x)$.

$$\lim_{x \to -2} f(x) = \underbrace{\lim_{x \to -2} -7x^2 + 13x - 25}_{\substack{\text{limit of} \\ \text{polynomial}}} \underset{\text{Theorem 3}}{=} \underbrace{-7(-2)^2 + 13(-2) - 25}_{\text{can substitute } x=-2} = -79.$$

Observe that we got the same number that we *estimated* in part (B), but there was a lot less work involved, and this time, we know that the result is correct.

End of [Example 1]

[Example 2] Let $f(x) = \sqrt{24 + x^2}$

(A) Find $f(5)$

Solution:

$$f(5) = \sqrt{24 + (5)^2} = \sqrt{24 + 25} = \sqrt{49} = 7.$$

(B) Find $\lim_{x \to 5} f(x)$

Solution:

$$\lim_{x \to 5} f(x) = \underbrace{\lim_{x \to 5} \sqrt{24 + x^2}}_{\text{limit outside the radical}} \underset{\text{Theorem 2.8}}{=} \underbrace{\sqrt{\underbrace{\lim_{x \to 5}(24 + x^2)}_{\text{limit of polynomial}}}}_{\text{limit inside the radical}} \underset{\text{Thm 3}}{=} \sqrt{\underbrace{24 + (5)^2}_{\text{can substitute } x=5}} = \sqrt{49} = 7.$$

[**Example 3**] Let $f(x) = \underbrace{\frac{x^2 - 6x + 5}{x^2 - 8x + 15}}_{\text{standard form}} = \underbrace{\frac{(x-1)(x-5)}{(x-3)(x-5)}}_{\text{factored form}}$

Observe that $f(x)$ is a *rational function* (a *ratio* of polynomials).

(A) Find $f(1)$

Solution: Although most students may be most familiar with the *standard form* of rational functions, it is the *factored form* that often be most useful when computing y values.

$$f(1) = \frac{\big((1)-1\big)\big((1)-5\big)}{\big((1)-3\big)\big((1)-5\big)} = \frac{(0)(-4)}{(-2)(-4)} = \frac{0}{8} = 0.$$

(B) Find $\lim_{x \to 1} f(x)$

Solution:

Again, although most students may be most familiar with the *standard form* of rational functions, it is the *factored form* that will be most useful when finding the limit.

$$\lim_{x\to 1} f(x) = \underbrace{\lim_{x\to 1} \frac{(x-1)(x-5)}{(x-3)(x-5)}}_{\substack{\text{limit of a rational} \\ \text{function that has} \\ x=1 \text{ in its domain}}} \underset{\text{Thm 3}}{=} \underbrace{\frac{\big((1)-1\big)\big((1)-5\big)}{\big((1)-3\big)\big((1)-5\big)}}_{\text{can substitute } x=1} = \frac{(0)(-4)}{(-2)(-4)} = \frac{0}{8} = 0.$$

(C) Find $f(3)$

Solution:

$$f(3)=\frac{\big((3)-1\big)\big((3)-5\big)}{\big((3)-3\big)\big((3)-5\big)}=\frac{(2)(-2)}{(0)(-2)}=\frac{-4}{0}\ \text{Does Not Exist.}$$

(D) Find $\lim_{x\to 3} f(x)$

Solution:

Notice that the limit of the numerator by itself is

$$\lim_{x\to 3} numerator=\underbrace{\lim_{x\to 3}(x-1)(x-5)}_{\text{limit of polynomial}}\underset{\text{Thm 3}}{=}\underbrace{\big((3)-1\big)\big((3)-5\big)}_{\text{can substitute }x=3}=(2)(-2)=-4\neq 0$$

Notice that the limit of the denomator by itself is

$$\lim_{x\to 3} denominator=\underbrace{\lim_{x\to 3}(x-3)(x-5)}_{\text{limit of polynomial}}\underset{\text{Thm 3}}{=}\underbrace{\big((3)-3\big)\big((3)-5\big)}_{\text{can substitute }x=3}=(0)(-2)=0$$

Therefore, Theorem 4 tells us

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{(x-1)(x-5)}{(x-3)(x-5)}\underset{\text{Theorem 4}}{=}\ \text{Does Not Exist.}$$

End of [Example 3]

End of Video