

# **Limits Video C: Limits of Piecewise-Defined Functions**

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## **Topics in this Video**

- Review the notation for piecewise-defined functions.
- Express the Absolute Value as a piecewise-defined function.
- Two examples of limits of piecewise-defined functions.

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## Useful Definitions from *Limits Video A*

### The Definition of *Limit*

**Symbol:**  $\lim_{x \rightarrow c} f(x) = L.$

**Spoken:** “The limit, as  $x$  approaches  $c$ , of  $f(x)$  is  $L$ .”

**Less-Abbreviated Symbol:**  $f(x) \rightarrow L$  as  $x \rightarrow c$ .

**Spoken:** “ $f(x)$  approaches  $L$  as  $x$  approaches  $c$ .”

**Usage:**  $x$  is a variable,  $f$  is a function,  $c$  is a real number, and  $L$  is a real number.

**Meaning:** as  $x$  gets closer and closer to  $c$ , but not equal to  $c$ , the value of  $f(x)$  gets closer and closer to  $L$  (may actually equal  $L$ ).

**Graphical Significance:** The graph of  $f$  appears to be heading for location  $(x, y) = (c, L)$  from both sides.

And note the difference between the symbols  $f(c)$  and  $\lim_{x \rightarrow c} f(x)$ .

- The symbol  $f(c)$  denotes the  $y$  value at the  $x$  value  $x = c$ .
- The symbol  $\lim_{x \rightarrow c} f(x)$  tells us about the *trend* in the  $y$  values as  $x$  gets closer and closer to  $c$ .

### **The Definition of *Limit from the Left***

**Symbol:**  $\lim_{x \rightarrow c^-} f(x) = L.$

**Spoken:** “The limit, as  $x$  approaches  $c$  from the left, of  $f(x)$  is  $L.$ ”

**Meaning:** as  $x$  gets closer and closer to  $c$ , but less than  $c$ , the value of  $f(x)$  gets closer and closer to  $L$  (may actually equal  $L$ ).

**Graphical Significance:** The graph of  $f$  appears to be heading for location  $(x, y) = (c, L)$  from the left.

### **The Definition of *Limit from the Right***

**Symbol:**  $\lim_{x \rightarrow c^+} f(x) = L.$

**Spoken:** “The limit, as  $x$  approaches  $c$  from the right, of  $f(x)$  is  $L.$ ”

**Meaning:** as  $x$  gets closer and closer to  $c$ , but greater than  $c$ , the value of  $f(x)$  gets closer and closer to  $L$  (may actually equal  $L$ ).

**Graphical Significance:** The graph of  $f$  appears to be heading for location  $(x, y) = (c, L)$  from the right.

**The Definition of *Limit* written as a 3-part test involving *One-Sided Limits*.**

**Symbol:**  $\lim_{x \rightarrow c} f(x) = L.$

**Meaning:** The function passes this three-part test

**test a:** The limit from the left,  $\lim_{x \rightarrow c^-} f(x)$ , exists

**test b:** The limit from the right,  $\lim_{x \rightarrow c^+} f(x)$ , exists

**test c:** The values of the limits from the left and right match, with value  $L$ . That is,

$$\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x).$$

## Useful Properties of Limits from *Limits Video B*

### **THEOREM 2** Properties of Limits

Let  $f$  and  $g$  be two functions, and assume that

$$\lim_{x \rightarrow c} f(x) = L \quad \lim_{x \rightarrow c} g(x) = M$$

where  $L$  and  $M$  are real numbers (both limits exist). Then

1.  $\lim_{x \rightarrow c} k = k$  for any constant  $k$
2.  $\lim_{x \rightarrow c} x = c$
3.  $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = L + M$
4.  $\lim_{x \rightarrow c} [f(x) - g(x)] = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x) = L - M$
5.  $\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x) = kL$  for any constant  $k$
6.  $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = [\lim_{x \rightarrow c} f(x)][\lim_{x \rightarrow c} g(x)] = LM$
7.  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{L}{M}$  if  $M \neq 0$
8.  $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)} = \sqrt[n]{L}$  if  $L > 0$  or  $n$  is odd

### **THEOREM 3** Limits of Polynomial and Rational Functions

1.  $\lim_{x \rightarrow c} f(x) = f(c)$  for  $f$  any polynomial function.
2.  $\lim_{x \rightarrow c} r(x) = r(c)$  for  $r$  any rational function with a nonzero denominator at  $x = c$ .

### **THEOREM 4** Limit of a Quotient

If  $\lim_{x \rightarrow c} f(x) = L$ ,  $L \neq 0$ , and  $\lim_{x \rightarrow c} g(x) = 0$ ,

then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} \quad \text{does not exist}$$

**In this video, we will be continuing an *Analytical Approach* to Limits  
(That is, the function  $f(x)$  is described by a *formula*, not a graph.)**

**We will study two examples involving *piecewise-defined* functions**

**[Example 1]** Let  $f(x) = \begin{cases} -2x + 10, & x \leq 3; \\ x^2, & x > 3. \end{cases}$

(A) Find  $f(3)$ .

(B) Find  $\lim_{x \rightarrow 3^-} f(x)$ .

(C) Find  $\lim_{x \rightarrow 3^+} f(x)$ .

(D) Find  $\lim_{x \rightarrow 3} f(x)$ .

(E) Illustrate with a graph.

**Solution:**

We must start by making sure that we understand what the notation for the function  $f(x)$  means. This function is an example of a *piecewise defined* function. That is, it is described by different formulas on different pieces of its domain. The notation is very much abbreviated. To understand how the function works, it is helpful to think of it in a less abbreviated form.

$$f(x) = -2x + 10 \text{ when } x \leq 3.$$

$$f(x) = x^2 \text{ when } x > 3.$$

It is even more helpful to think of the function in a form that reflects how one *uses* the formula for a piecewise defined function. Given an input  $x$ , one first determines which piece of the domain  $x$  is in, and then uses the appropriate formula to compute the output. This process would be better described with the following two sentences

If  $x \leq 3$ , then  $f(x)$  is computed using the formula  $f(x) = -2x + 10$ .

If  $x > 3$ , then  $f(x)$  is computed using the formula  $f(x) = x^2$ .

Now we are ready to do the solve the problem.

(A) To find  $f(3)$ , we note that because  $3 \leq 3$  is true, we use the formula  $f(x) = -2x + 10$  to compute the output.

$$f(3) \underset{\substack{= \\ \text{use} \\ \text{appropriate} \\ \text{formula}}}{=} -2(3) + 10 = 4.$$

(B) To find the limit from the left,  $\lim_{x \rightarrow 3^-} f(x)$ , we must first note that the symbol  $x \rightarrow 3^-$  tells us that we should consider  $x$  values that are getting closer and closer to 3 but less than 3. For these  $x$  values, the formula for  $f(x)$  is  $f(x) = -2x + 10$ .

$$\lim_{x \rightarrow 3^-} f(x) \underset{\substack{\text{use} \\ \text{appropriate} \\ \text{formula}}}{=} \lim_{x \rightarrow 3^-} \underbrace{-2x + 10}_{\substack{\text{limit of a} \\ \text{polynomial}}} \underset{\text{Theorem 3}}{=} \underbrace{-2(3) + 10}_{\substack{\text{can substitute } x=3}} = 4.$$

(C) To find the limit from the right,  $\lim_{x \rightarrow 3^+} f(x)$ , we note that the symbol  $x \rightarrow 3^+$  tells us that we should consider  $x$  values that are getting closer and closer to 3 but greater than 3.

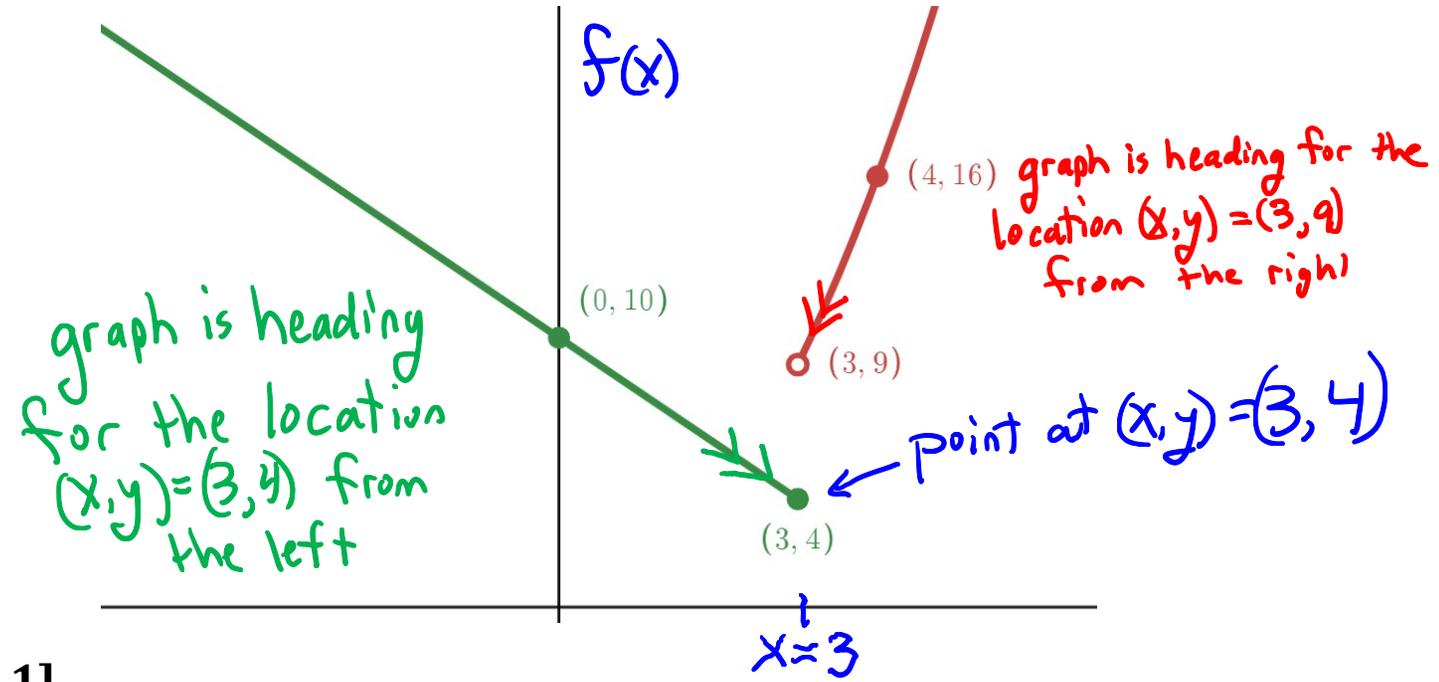
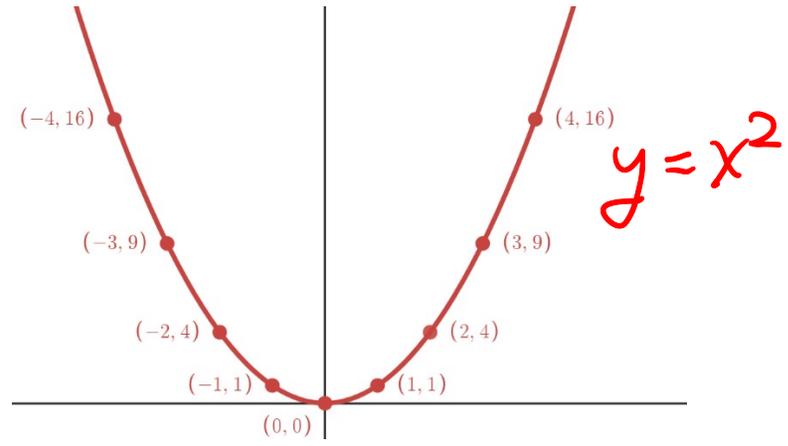
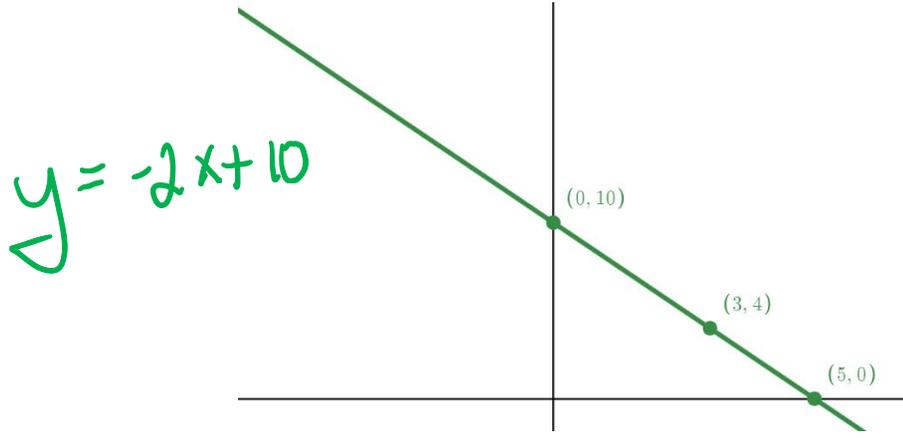
For these  $x$  values, the formula for  $f(x)$  is  $f(x) = x^2$ .

$$\lim_{x \rightarrow 3^+} f(x) \underset{\substack{\text{use} \\ \text{appropriate} \\ \text{formula}}}{=} \lim_{x \rightarrow 3^+} x^2 \underset{\text{Theorem 3}}{=} \underbrace{(3)^2}_{\text{can substitute } x=3} = 9.$$

(D) Finding the two-sided limit  $\lim_{x \rightarrow 3} f(x)$  is straightforward.

$\lim_{x \rightarrow 3} f(x)$  does not exist because the left and right limits don't match.

(E) To draw the graph of  $f(x)$ , it is helpful to first draw separate graphs for each of the formulas that make up the description of  $f(x)$ , and then use these separate graphs to build a graph of  $f(x)$ .



End of [Example 1]

**[Example 2]** Let  $f(x) = \frac{x - 11}{|x - 11|}$ .

(A) Find  $f(11)$ .

(B) Find  $\lim_{x \rightarrow 11^-} f(x)$ .

(C) Find  $\lim_{x \rightarrow 11^+} f(x)$ .

(D) Find  $\lim_{x \rightarrow 11} f(x)$ .

(E) Illustrate with a graph.

**Solution:** As in the previous example, we must start by making sure that we understand what the notation means.

You are familiar with the behavior of the absolute value function when the thing inside is a *number*. For example,

$$|5| = 5.$$

$$|-7| = 7.$$

$$|0| = 0.$$

But you are probably not so familiar with the absolute value function in abstract settings, where the thing inside the absolute value involves a *variable*. The absolute value symbol actually represents a piecewise-defined function. Here is the definition:

### **Definition of the Absolute Value Function**

$$|x| = \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

As in **[Example 1]**, it is helpful to think of the function in a form that reflects how one *uses* the formula.

### **Definition of the Absolute Value Function, in Sentences**

If  $x \geq 0$ , then the absolute value is computed using the formula  $|x| = x$ .

If  $x < 0$ , then the absolute value is computed using the formula  $|x| = -x$ .

In order to understand how the absolute value function is used in when the thing inside is a more complicated expression, it is helpful to consider what I call the *empty version* of the function.

**Definition of the Absolute Value Function, in Sentences, *Empty Version***

If  $( ) \geq 0$ , then the absolute value is computed using the formula  $|( )| = ( )$ .

If  $( ) < 0$ , then the absolute value is computed using the formula  $|( )| = -( )$ .

In other words, the absolute value symbol can mean different things, depending on whether the thing inside it is known to be positive or negative or zero.

Because the absolute value symbol actually represents a piecewise-defined function, the given one-line formula  $f(x) = \frac{x-11}{|x-11|}$  also represents a piecewise-defined function. Let's figure out how it behaves

- If  $x > 11$ , then  $x - 11 > 0$ , and so  $|x - 11| = x - 11$ .

In this case the expression for  $f(x) = \frac{x-11}{|x-11|}$  simplifies to

$$f(x) = \frac{x - 11}{|x - 11|} \stackrel{\substack{\text{use} \\ \text{appropriate} \\ \text{formula}}}{=} \frac{x - 11}{x - 11} \stackrel{\substack{\text{can cancel} \\ \text{because} \\ x-11 \neq 0}}{=} 1.$$

- If  $x = 11$ , then  $x - 11 = 0$ , and so  $|x - 11| = |0| = 0$

In this case the expression for  $f(x) = \frac{x-11}{|x-11|}$  simplifies to  $f(11) = \frac{11-11}{|11-11|} = \frac{0}{0}$ , which is undefined.

- If  $x < 11$ , then  $x - 11 < 0$ , and so  $|x - 11| = -(x - 11)$ .

In this case the expression for  $f(x)$  simplifies to

$$f(x) = \frac{x - 11}{|x - 11|} \stackrel{\substack{\text{use} \\ \text{appropriate} \\ \text{formula}}}{=} \frac{x - 11}{-(x - 11)} \stackrel{\substack{\text{can cancel} \\ \text{because} \\ x-11 \neq 0}}{=} -1.$$

Summarizing what we have just found, a description of  $f(x)$  in sentences is:

If  $x > 11$ , then  $f(x) = 1$ .

If  $x = 11$ , then  $f(x)$  is undefined.

If  $x < 11$ , then  $f(x) = -1$ .

A piecewise presentation of  $f(x)$  would be:

$$f(x) = \frac{x - 11}{|x - 11|} = \begin{cases} 1 & \text{when } x > 11, \\ \text{DNE} & \text{when } x = 11, \\ -1 & \text{when } x < 11. \end{cases}$$

Now we are ready to solve the problem

(A) Find  $f(11)$ .

**Solution:**

$$f(11) = \frac{11 - 11}{|11 - 11|} = \frac{0}{0}, \text{ which is } \textit{undefined}.$$

(B) Find  $\lim_{x \rightarrow 11^-} f(x)$ .

**Solution:**

To find  $\lim_{x \rightarrow 11^-} f(x)$ , we must first note that the symbol  $x \rightarrow 11^-$  tells us that we should consider  $x$  values that are getting closer and closer to 11 but less than 11. For these  $x$  values, the formula for  $f(x)$  is

$$f(x) = -1.$$

So the limit from the left is

$$\lim_{x \rightarrow 11^-} f(x) \stackrel{\substack{= \\ \text{use} \\ \text{appropriate} \\ \text{formula}}}{=} \underbrace{\lim_{x \rightarrow 11^-} -1}_{\substack{\text{limit of a} \\ \text{constant}}} \stackrel{\text{Thm 2.1}}{=} \underbrace{-1}_{\substack{\text{the constant}}} .$$

(C) Find  $\lim_{x \rightarrow 11^+} f(x)$ .

**Solution:**

The symbol  $x \rightarrow 11^+$  tells us that we should consider  $x$  values that are getting closer and closer to 11 but greater than 11. For these  $x$  values, the formula for  $f(x)$  is

$$f(x) = 1.$$

So the limit from the right is

$$\lim_{x \rightarrow 11^+} f(x) \stackrel{\substack{= \\ \text{use} \\ \text{appropriate} \\ \text{formula}}}{=} \lim_{x \rightarrow 11^+} \underbrace{1}_{\substack{\text{limit of a} \\ \text{constant}}} \stackrel{\text{Thm 2.1}}{=} \underbrace{1}_{\substack{\text{the constant}}} .$$

(D) Find  $\lim_{x \rightarrow 11} f(x)$ .

**Solution:**

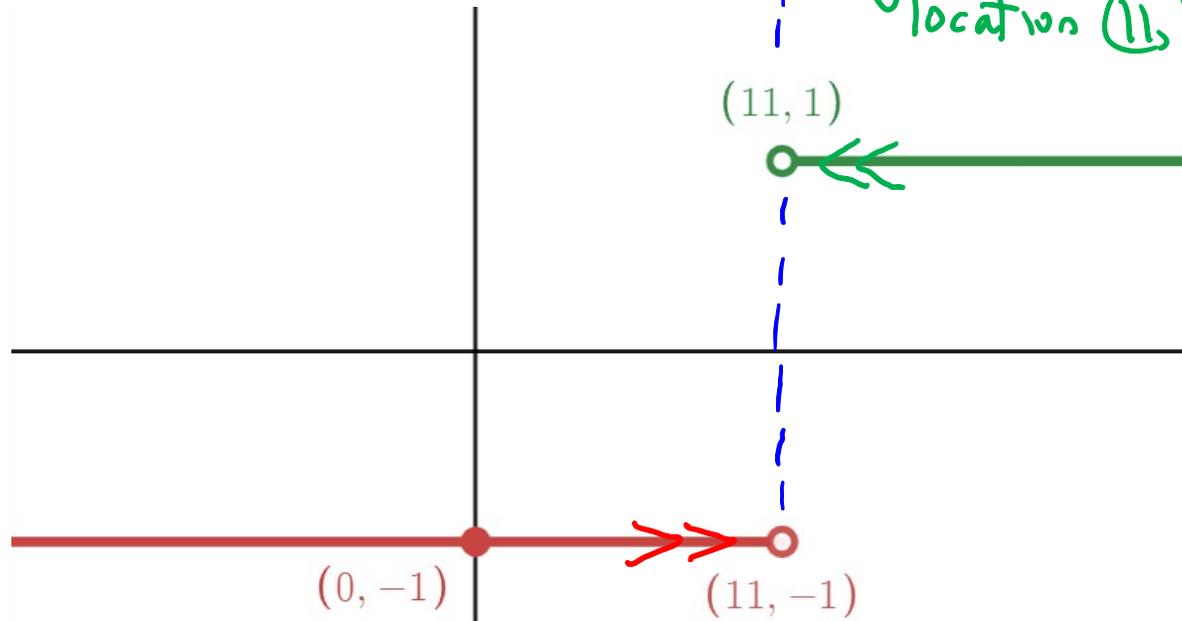
The limit does not exist, because the left and right limits don't match.

(E) Illustrate with a graph.

**Solution:** The formula for the function is  $f(x) = \frac{x-11}{|x-11|}$ , but the sentence description is more helpful for making the graph.

- If  $x > 11$ , then  $f(x) = 1$ .
- If  $x = 11$ , then  $f(x)$  is undefined.
- If  $x < 11$ , then  $f(x) = -1$ .

graph has no y value at  $x=11$   
graph is heading for the location  $(11, 1)$  from the right.



$y = -1$

$(11, 1)$

$y = 1$

$(0, -1)$

$(11, -1)$

End of [Example 2]

graph is heading for the location  $(x, y) = (11, -1)$  from the left.

End of Video