

# **Limits Video D: Limits That Involve Cancelling Inside the Limit**

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## **Topics in this Video**

- Ratios of Numbers
- The most important concept for the first month of Calculus
- Limits of Ratios
- Indeterminate Forms
- Example of computing  $y$  values and limits for a rational function
- Common Mistakes, and the Importance of using proper limit notation
- (Printed Notes Only) Another example of computing  $y$  values and limits for a rational function

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## Useful Definition from *Limits Video A*

### The Definition of *Limit*

**Symbol:**  $\lim_{x \rightarrow c} f(x) = L.$

**Spoken:** “The limit, as  $x$  approaches  $c$ , of  $f(x)$  is  $L$ .”

**Less-Abbreviated Symbol:**  $f(x) \rightarrow L$  as  $x \rightarrow c$ .

**Spoken:** “ $f(x)$  approaches  $L$  as  $x$  approaches  $c$ .”

**Usage:**  $x$  is a variable,  $f$  is a function,  $c$  is a real number, and  $L$  is a real number.

**Meaning:** as  $x$  gets closer and closer to  $c$ , but not equal to  $c$ , the value of  $f(x)$  gets closer and closer to  $L$  (may actually equal  $L$ ).

**Graphical Significance:** The graph of  $f$  appears to be heading for location  $(x, y) = (c, L)$  from both sides.

And note the difference between the symbols  $f(c)$  and  $\lim_{x \rightarrow c} f(x)$ .

- The symbol  $f(c)$  denotes the  $y$  value at the  $x$  value  $x = c$ .
- The symbol  $\lim_{x \rightarrow c} f(x)$  tells us about the *trend* in the  $y$  values as  $x$  gets closer and closer to  $c$ .

## Useful Properties of Limits from *Limits Video B*

### **THEOREM 3** Limits of Polynomial and Rational Functions

1.  $\lim_{x \rightarrow c} f(x) = f(c)$  for  $f$  any polynomial function.
2.  $\lim_{x \rightarrow c} r(x) = r(c)$  for  $r$  any rational function with a nonzero denominator at  $x = c$ .

### **THEOREM 4** Limit of a Quotient

If  $\lim_{x \rightarrow c} f(x) = L$ ,  $L \neq 0$ , and  $\lim_{x \rightarrow c} g(x) = 0$ ,

then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} \quad \text{does not exist}$$

**In this video, we will be continuing an *Analytical Approach to Limits*  
(That is, the function  $f(x)$  is described by a *formula*, not a graph.)**

We will study Examples involving cancelling inside the limit.

## Ratios of Numbers

Recall the possible quotients of *zero* and *non-zero real numbers*, and their resulting values as determined by the rules of arithmetic.

	<i>numerator</i> $\neq$ 0	<i>numerator</i> = 0
<i>denominator</i> $\neq$ 0	$\frac{\textit{nonzero}}{\textit{nonzero}} = \textit{nonzero}$	$\frac{0}{\textit{nonzero}} = 0$
<i>denominator</i> = 0	$\frac{\textit{nonzero}}{0} \textit{ DNE}$	$\frac{0}{0} \textit{ DNE}$

## Most Important Concept of the First Month of Calculus

A key concept in this video is the most important concept of the first month of Calculus:

*When can one cancel terms, and why?*

Observe that

- We can cancel a quotient of like terms if they are non-zero. For example,  $\frac{5}{5} = 1$ .
- But we cannot cancel  $\frac{0}{0}$ . The expression  $\frac{0}{0}$  does not equal 1. (It also does not equal 0.) The expression  $\frac{0}{0}$  does not define a real number. We say that  $\frac{0}{0}$  is *undefined*, or that it *does not exist (DNE)*.
- Without knowing anything about the value of the variable  $x$ , we cannot cancel  $\frac{x}{x}$ . (If it turns out that  $x \neq 0$ , then  $\frac{x}{x} = 1$ . But if it turns out that  $x = 0$ , then  $\frac{x}{x}$  is *undefined*. So we can't say in advance.)
- But if we are given extra information enables us to determine that  $x \neq 0$  or  $x = 0$ , then we can replace the expression  $\frac{x}{x}$  with either a 1 or the expression *DNE*.

## Limits of Ratios

Recall what we know so far about computing the limit of quotients. That is, limits of the form

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}.$$

There are two cases that we have considered so far.

### One Case: When the the limit of the denominator is not zero.

Theorem 2.7 tells us that

$$\text{If } \lim_{x \rightarrow c} f(x) = L \text{ and if } \lim_{x \rightarrow c} g(x) = M \neq 0,$$

*then the limit of the quotient is equal to the quotient of the limits.*

That is,

$$\text{If } \lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M \neq 0,$$

$$\text{then } \underbrace{\lim_{x \rightarrow c} \frac{f(x)}{g(x)}}_{\text{limit of quotient with limit of denominator not zero}} \stackrel{\text{Theorem 2.7}}{=} \underbrace{\frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}}_{\text{quotient of limits}} = \frac{L}{M}.$$

**Another Case: When the limit of the numerator is not zero and the limit of the denominator is zero.**

Theorem 4 tells us that

$$\text{If } \lim_{x \rightarrow c} f(x) = L \neq 0 \text{ and if } \lim_{x \rightarrow c} g(x) = M = 0,$$

*then the limit of the quotient does not exist.*

That is,

$$\text{If } \lim_{x \rightarrow c} f(x) = L \neq 0 \text{ and } \lim_{x \rightarrow c} g(x) = M = 0,$$

$$\text{then } \lim_{x \rightarrow c} \frac{f(x)}{g(x)} \text{ DNE.}$$



## Indeterminate Forms

**New Case: When the limits of both the numerator and the denominator are zero.**

A case that we have not yet considered is the case when the limit of the numerator and the limit of the denominator are both zero. It turns out that if both  $\lim_{x \rightarrow c} f(x) = L = 0$  and  $\lim_{x \rightarrow c} g(x) = M = 0$ , then it is impossible to say immediately whether or not the limit of the ratio  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  exists and, if it does exist, what its value will be. Some more steps need to be done in order to convert the limit into one of the forms for which our theorems about limits (Theorems 2,3,4) can be used. Only then can a determination be made about the existence and value of the limit. For that reason, limits of type  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ , where  $\lim_{x \rightarrow c} f(x) = L = 0$  and  $\lim_{x \rightarrow c} g(x) = M = 0$ , are said to be in *indeterminate form*.

## DEFINITION Indeterminate Form

If  $\lim_{x \rightarrow c} f(x) = 0$  and  $\lim_{x \rightarrow c} g(x) = 0$ , then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form**.

The previous paragraph says, rather vaguely, that

*“...Some more steps need to be done in order to convert the limit into one of the forms for which our theorems about limits (Theorems 2,3,4) can be used. Only then can a determination be made about the existence and value of the limit....”*

What exactly does that mean? What *steps*?

The steps involve cancelling terms in the numerator and denominator. Remember that at the start of this video, it was mentioned that that operation, cancelling terms, involves the most important concept of the first month of calculus:

*When can one cancel terms, and why?*

We will see that when we encounter a limit that is an *indeterminate form*, we will be doing steps involving *cancelling terms*, in order to convert the limit to a new form that is *not* an indeterminate form. Then, we will be able to apply one of our theorems about limits (*Theorems 2,3,4*) to determine if the limit exists and, if it does exist, to give its value.

The various types of limits of quotients are summarized in the table below

	$\lim_{x \rightarrow c} f(x) = L \neq 0$	$\lim_{x \rightarrow c} f(x) = 0$
$\lim_{x \rightarrow c} g(x) = M \neq 0$	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M} \neq 0$ by <i>Theorem 2.7.</i>	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{0}{M} = 0$ by <i>Theorem 2.7.</i>
$\lim_{x \rightarrow c} g(x) = 0$	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} \text{ DNE}$ by <i>Theorem 4.</i>	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} \text{ ???}$ <i>indeterminate form</i>

For the rest of this video, we will consider examples of computing  $y$  values and limits for quotients. Some of the limits will involve *indeterminate forms*; some will not.

**[Example 1]** Let  $f(x) = \frac{x^2 - 2x - 3}{x - 3} = \frac{(x - 3)(x + 1)}{(x - 3)}$

**(A) (Computing y values.)** Find  $f(c)$  for  $c = 1, 2, 3$

**Solution:** The function  $f(x)$  is called a *rational function* because it is a *ratio of polynomials*.

The two forms shown are called the *standard form* and the *factored form*. Although the *standard form* of polynomials and rational functions may be a more familiar form, when computing  $y$  values, the *factored form* makes the computations easier.

**Computing the y value  $f(1)$**

$$f(1) = \frac{((1) - 3)((1) + 1)}{((1) - 3)} = \frac{\cancel{(-2)}(2)}{(-2)} = 2.$$

We can cancel  $\frac{-2}{-2}$  because  $-2 \neq 0$ .

## Computing the $y$ value $f(2)$

$$f(2) = \frac{((2) - 3)((2) + 1)}{((2) - 3)} = \frac{(-1)(3)}{(-1)} = 3.$$

We can cancel  $\frac{-1}{-1}$  because  $-1 \neq 0$ .

## Computing the $y$ value $f(3)$

$$f(3) = \frac{((3) - 3)((3) + 1)}{((3) - 3)} = \frac{(0)(3)}{(0)} = \frac{0}{0}, \text{ DNE.}$$

We cannot cancel  $\frac{0}{0}$ , so  $f(3)$  *does not exist*. (This is a quotient of *real numbers*; it is *not* a *limit*.)

The quotient  $\frac{0}{0}$  is *not* an *indeterminate form*. There is no uncertainty about what this expression might equal. It is simply an expression that does not represent a real number. Therefore, we say that the expression is *undefined*, or *does not exist*.)

**(B) (Finding Limits.)** Find  $\lim_{x \rightarrow c} f(x)$  for  $c = 1, 2, 3$ .

**Solution:** As was the case in part (A), although the *standard form* of polynomials and rational functions may be a more familiar form, when computing the limit as  $x \rightarrow c$ , the *factored form* makes computations easier.

**Computing the limit as  $x \rightarrow 1$**

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{(x-3)(x+1)}{(x-3)} \stackrel{\text{Theorem 3}}{=} \frac{((1)-3)((1)+1)}{((1)-3)} = \frac{\cancel{(-2)}(2)}{\cancel{(-2)}} \stackrel{\text{can cancel because } \underline{-2 \neq 0}}{=} 2$$

We are allowed to use Theorem 3 because substituting in  $x = 1$  does not cause the denominator to be zero.

## Computing the limit as $x \rightarrow 2$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{(x-3)(x+1)}{(x-3)} \stackrel{\text{Theorem 3}}{=} \frac{((2)-3)((2)+1)}{((2)-3)} = \frac{(-1)(3)}{(-1)} \stackrel{\substack{\text{can cancel} \\ \text{because} \\ -1 \neq 0}}{=} 3$$

limit of rational function  
with  $x=2$  in its domain

can substitute  $x=2$

We are allowed to use Theorem 3 because substituting in  $x = 2$  does not cause the denominator to be zero.



## Computing the limit as $x \rightarrow 3$

We first investigate what happens if we try substituting  $x = 3$  into the expression

$$\frac{(x - 3)(x + 1)}{(x - 3)} \xrightarrow{\text{try substituting } x=3} \frac{((3) - 3)((3) + 1)}{((3) - 3)} = \frac{(0)(4)}{(0)} = \frac{0}{0}$$

Because we reach the expression  $\frac{0}{0}$ , we know that **we should not substitute  $x = 3$ !**

(Remember that none of the Limit Properties allow us to substitute an  $x$  value into an expression if it results in a denominator of 0.)

But what are we to make of the fact that when we substitute  $x = 3$  into the expression, we reach the expression  $\frac{0}{0}$ ??

Observe that simply substituting  $x = 3$  into the expression really amounts to taking the limit of the numerator and the limit of the denominator separately. To help you understand what that means, I'll do those limits more explicitly on the next page.

Take the limits of the numerator and denominator separately.

- Limit of numerator is  $\lim_{x \rightarrow 3} \underbrace{(x - 3)(x + 1)}_{\text{limit of a polynomial}} \stackrel{\text{Thm 3}}{=} \underbrace{((3) - 3)((3) + 1)}_{\text{can substitute } x=3} = (0)(4) = 0.$

- Limit of denominator is  $\lim_{x \rightarrow 3} \underbrace{(x - 3)}_{\text{limit of a polynomial}} \stackrel{\text{Thm 3}}{=} \underbrace{((3) - 3)}_{\text{can substitute } x=3} = 0.$

Since the limits of the numerator and denominator by themselves are both zero, we realize that the limit is an *indeterminate form*.

### **DEFINITION Indeterminate Form**

If  $\lim_{x \rightarrow c} f(x) = 0$  and  $\lim_{x \rightarrow c} g(x) = 0$ , then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form**.

Remember that when one encounters a limit that is an *indeterminate form*,

*“...Some more steps need to be done in order to convert the limit into one of the forms for which our theorems about limits (Theorems 2,3,4) can be used. Only then can a determination be made about the existence and value of the limit....”.*

The steps involve *cancelling terms* in the numerator and denominator. Remember that cancelling terms involves the most important concept of the first month of calculus:

***When can one cancel terms, and why?***

I'll do the limit in detail on the next page.

Here is a presentation of the limit, with justifications for each step.

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \underbrace{\frac{(x-3)(x+1)}{(x-3)}}_{\text{indeterminate form}} \stackrel{\substack{= \\ \text{can cancel} \\ \text{because} \\ x-3 \neq 0}}}{=} \underbrace{\lim_{x \rightarrow 3} (x+1)}_{\substack{\text{limit of polynomial} \\ \text{(not indeterminate)}}} \stackrel{\text{Theorem 3}}{=} \underbrace{((3)+1)}_{\text{can substitute } x=3} = 4.$$

Justification for the Cancellation Step:

since  $x \rightarrow 3$ ,

we know that  $x \neq 3$ ,

so we know that  $x - 3 \neq 0$ ,

so we know that we can cancel  $\frac{(x-3)}{(x-3)}$ .

Observe that the limit proceeded in the manner that was described a few pages ago:

*We will see that when we encounter a limit that is an indeterminate form, we will be doing steps involving cancelling terms, in order to convert the limit to a new form that is not an indeterminate form. Then, we will be able to apply one of our theorems about limits (Theorems 2,3,4) to determine if the limit exists and, if it does exist, to give its value.*

Let's summarize our results of parts (A) and (B), and consider what they tell us about the graph of

$$f(x) = \frac{x^2 - 2x - 3}{x - 3} = \frac{(x - 3)(x + 1)}{(x - 3)}.$$

**At  $x = 1$**

- In part (A), we found that  $f(1) = 2$ .

This tells us that there is a point on the graph with coordinates  $(x, y) = (1, 2)$ .

- In part (B), we found that  $\lim_{x \rightarrow 1} f(x) = 2$ .

This tells us that the graph is heading for the location  $(x, y) = (1, 2)$ .

**At  $x = 2$**

- In part (A), we found that  $f(2) = 3$ .

This tells us that there is a point on the graph with coordinates  $(x, y) = (2, 3)$ .

- In part (B), we found that  $\lim_{x \rightarrow 2} f(x) = 3$ .

This tells us that the graph is heading for the location  $(x, y) = (2, 3)$ .

### At $x = 3$

- In part (A), we found that  $f(3)$  does not exist.

This tells us that there is no point on the graph with  $x$  coordinate  $x = 3$ .

- In part (B), we found that  $\lim_{x \rightarrow 3} f(x) = 4$ .

This tells us that the graph is heading for the location  $(x, y) = (3, 4)$ .

- Since the graph is heading for the location  $(x, y) = (3, 4)$ , but there is no point on the graph with  $x$  coordinate  $x = 3$ , we realize there is a *hole in the graph* at the location  $(x, y) = (3, 4)$ .

The calculations that we did at  $x = 3$ , finding that  $f(3)$  does not exist while  $\lim_{x \rightarrow 3} f(x)$  does exist, illustrate the most important concept of the first month of Calculus, mentioned at the start of this video.

***When can one cancel terms, and why?***

We *were not* allowed to cancel terms when computing  $f(3)$ , while we *were* allowed to cancel terms when computing  $\lim_{x \rightarrow 3} f(x)$ .

## Do these results make sense?

Consider the graphs of  $f(x) = \frac{x^2-2x-3}{x-3} = \frac{(x-3)(x+1)}{(x-3)}$  and a very similar function  $g(x) = x + 1$ .

Let's make tables of values for both functions

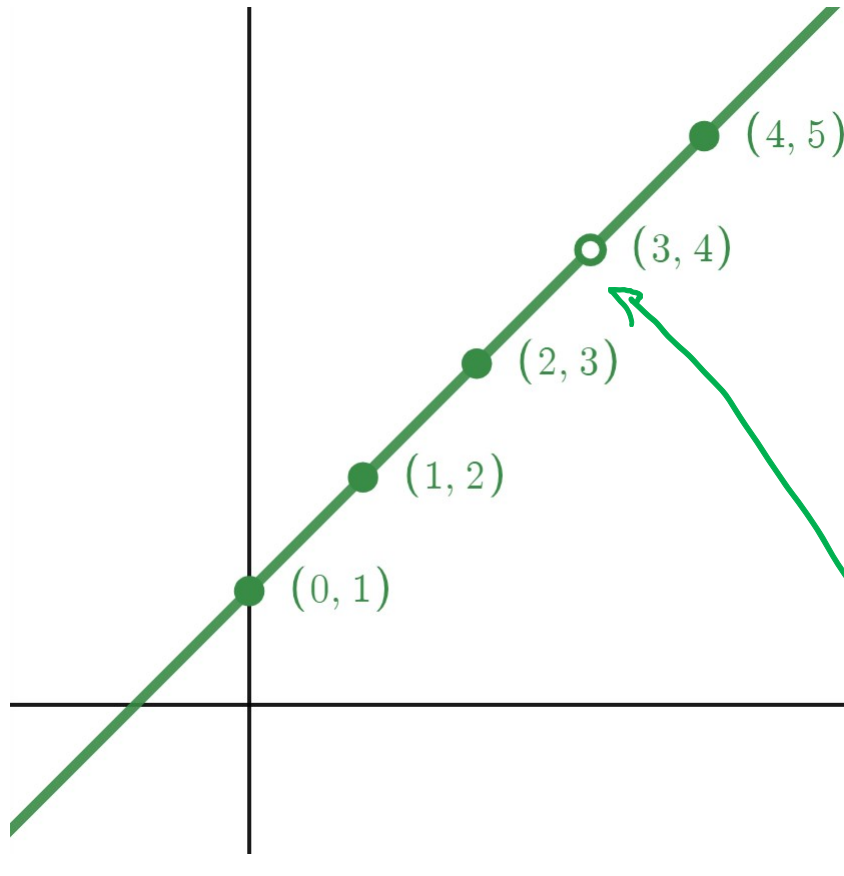
$x$	$f(x) = \frac{(x-3)(x+1)}{(x-3)}$
0	$f(0) = \frac{((0)-3)((0)+1)}{((0)-3)} = \frac{\cancel{(-3)}(1)}{\cancel{(-3)}} = 1.$
1	$f(1) = \frac{((1)-3)((1)+1)}{((1)-3)} = \frac{\cancel{(-2)}(2)}{\cancel{(-2)}} = 2.$
2	$f(2) = \frac{((2)-3)((2)+1)}{((2)-3)} = \frac{\cancel{(-1)}(3)}{\cancel{(-1)}} = 3.$
3	$f(3) = \frac{((3)-3)((3)+1)}{((3)-3)} = \frac{(0)(4)}{(0)} = \frac{0}{0} \text{ DNE.}$
4	$f(4) = \frac{((4)-3)((4)+1)}{((4)-3)} = \frac{\cancel{(1)}(5)}{\cancel{(1)}} = 5.$

$x$	$g(x) = x + 1$
0	$g(0) = (0) + 1 = 1.$
1	$g(1) = (1) + 1 = 2.$
2	$g(2) = 2 + 1 = 3.$
3	$g(3) = (3) + 1 = 4.$
4	$g(4) = (4) + 1 = 5.$

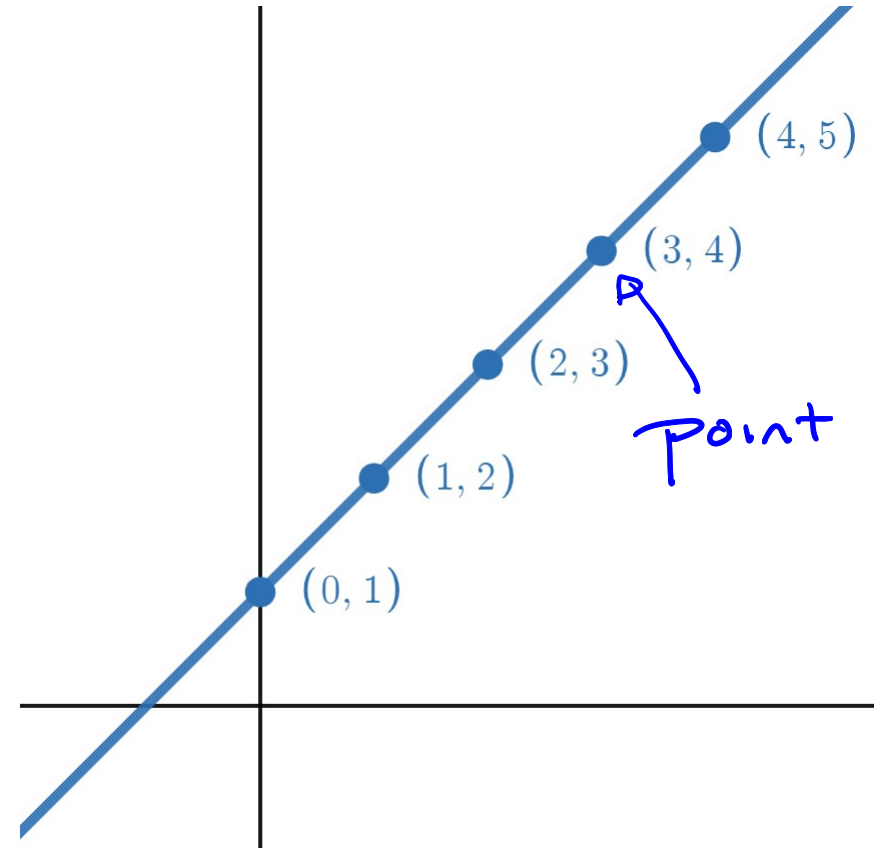
Observe that we were able to cancel terms of the form  $\frac{a}{a}$  only when  $a \neq 0$ .

Use this data to make graphs of  $f(x)$  and  $g(x)$  on separate axes.

$$f(x) = \frac{x^2 - 2x - 3}{x - 3} = \frac{(x - 3)(x + 1)}{(x - 3)}$$



$$g(x) = x + 1$$



Observe the hole in graph of  $f(x)$  at  $(x, y) = (3, 4)$ .

The hole is caused by the factors  $\frac{(x-3)}{(x-3)}$  in the formula for  $f(x)$ .

The hole in the graph of  $f(x)$  corresponds to the fact that  $f(3)$  DNE, but  $\lim_{x \rightarrow 3} f(x) = 4$ .



Realize that the above tables of data and the corresponding graphs tell us that  $f(x)$  and  $g(x)$  are *not* the same function. That is, as *functions*, these two expressions are *not equal*.

$$\frac{(x-3)(x+1)}{(x-3)} \neq x+1.$$

But the expressions do have the *same limit*

$$\lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{(x-3)} = \lim_{x \rightarrow 3} (x+1).$$

Remark: One must be careful to always write correct mathematical equations.

The following equation is not true!

~~$$\frac{(x-3)(x+1)}{(x-3)} = x+1.$$~~

But this equation is true!

$$\lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{(x-3)} = \lim_{x \rightarrow 3} (x+1).$$

**End of [Example 1]**

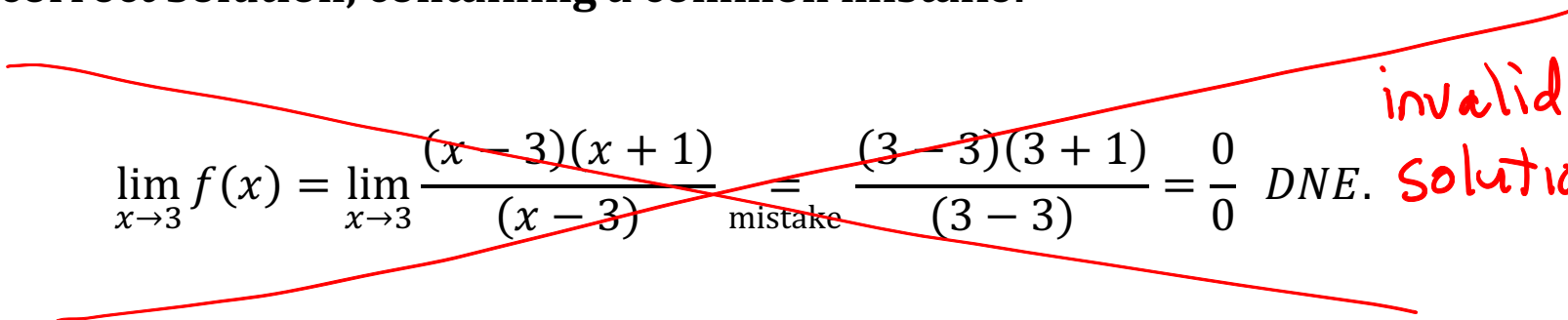
## Common Mistakes, and the Importance of using proper limit notation

Before continuing on to a second example, it is worth making note of two very common mistakes that students often make when computing limits.

Consider the following two “solutions” to the problem from **[Example 1] (B)**

$$\text{For } f(x) = \frac{(x-3)(x+1)}{(x-3)}, \text{ find } \lim_{x \rightarrow 3} f(x).$$

**Alice’s Incorrect Solution, containing a common mistake:**


$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{(x-3)} \stackrel{\text{mistake}}{=} \frac{(3-3)(3+1)}{(3-3)} = \frac{0}{0} \text{ DNE.}$$

*invalid solution!*

Realize that Alice made a mistake in the step indicated. The rules for computing limits (Theorems 2 and 3) only allow one to substitute  $x = c$  into a quotient if doing so *does not cause the denominator to be zero!*

As a result of her mistake, Alice got the wrong answer for the limit.

## Bob's Incorrect Solution, containing two common mistakes:

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{(x-3)} \underset{\text{mistake \#1}}{=} \frac{(3-3)(3+1)}{(3-3)} \underset{\text{mistake \#2}}{=} (3+1) = 4. \quad \text{invalid solution}$$

Observe that Bob got the right numerical answer, but he made two mistakes.

- Mistake #1: One is not allowed to substitute in  $x = 3$ , because it causes the denominator to become zero.
- Mistake #2: One cannot cancel  $\frac{0}{0}$  and replace it with the number 1.

So Bob's Solution is worse than Alice's solution, even though Bob got the right numerical answer.

## The importance of using proper limit notation.

Students sometimes leave out important notation, writing solutions such as the following:

$$\lim_{x \rightarrow 3} f(x) = \frac{(x-3)(x+1)}{(x-3)} = x+1 = 3+1 = 4$$

↑  
not equal

↑  
not equal

↑  
not equal

invalid solution

It is important to realize that although this line shows the correct value for the limit, the steps are incorrect. In fact, three of the equal signs are not valid!!

To understand why, consider carefully what the symbols on that line represent.

## First invalid equal sign

- The expression  $\lim_{h \rightarrow 3} f(x)$  represents a *number*. (We found in [Example 1] that it represents the number 4.)
- The expression  $\frac{(x-3)(x+1)}{(x-3)}$  is a *function* in the variable  $x$ .

A *number* is not the same thing as a *function* in the variable  $x$ , so the first equal sign is invalid!

$$\lim_{x \rightarrow 3} f(x) \neq \frac{(x-3)(x+1)}{(x-3)}$$

## Second invalid equal sign

- The expression  $\frac{(x-3)(x+1)}{(x-3)}$  is a *function* in the variable  $x$ , with domain all  $x \neq 3$ .
- The expression  $x + 1$  is a *function* in the variable  $x$ , with domain all real numbers  $x$ .

These functions are *not equal*, because they have *different domains*!

Therefore, the second equal sign is invalid!

$$\frac{(x-3)(x+1)}{(x-3)} \neq x+1$$

### Third invalid equal sign

- The symbol  $x + 1$  is a *function* in the variable  $x$ .
- The expression  $0 + 1$  is a *number*.

A *function* in the variable  $x$  is not the same thing as a *number*, so the third equal sign is invalid!

$$x + 1 \neq 3 + 1$$

### Conclusion about limit notation.

A correction solution to a limit problem should show not only the correct final answer for the limit, but also correct steps, with correct limit notation and explanation of the steps (as modeled in my solution to **[Example 1]**).

## Example to be Presented in the Notes but Not Discussed in the Video

The next example will be included in the printed notes accompanying this video, but I will not discuss the example in the video. (The concepts are the same concepts that were discussed in the previous example.) You should study the printed notes carefully on your own.

**[Example 2]** Let  $f(x) = \frac{x^2 - 6x + 5}{x^2 - 8x + 15} = \frac{(x - 1)(x - 5)}{(x - 3)(x - 5)}$ .

**(A) Computing y values.** Find  $f(c)$  for  $c = 1, 3, 5$

**(B) Finding Limits.** Find  $\lim_{x \rightarrow c} f(x)$  for  $c = 1, 3, 5$ .

**(C) Confirm Results with a Graph**



**(A) (Computing y values.)** Find  $f(c)$  for  $c = 1, 3, 5$

**Solution:** As was mentioned in **[Example 1]**, remember that when computing y values, the *factored form* makes the computations easier.

**Computing the y value  $f(1)$**

$$f(1) = \frac{((1) - 1)((1) - 5)}{((1) - 3)((1) - 5)} = \frac{(0)(-4)}{(-2)(-4)} = \frac{(0)}{(-2)} = 0.$$

We can cancel  $\frac{-4}{-4}$  because  $-4 \neq 0$ .

### Computing the $y$ value $f(3)$

$$f(3) = \frac{((3) - 1)((3) - 5)}{((3) - 3)((3) - 5)} = \frac{(2)(-2)}{(0)(-2)} = \frac{-4}{0} \text{ DNE.}$$

Division by 0 is undefined, so  $f(3)$  *does not exist*.

## Computing the $y$ value $f(5)$

$$f(5) = \frac{((5) - 1)((5) - 5)}{((5) - 3)((5) - 5)} = \frac{(4)(0)}{(2)(0)} = \frac{0}{0} \text{ DNE.}$$

We cannot cancel  $\frac{0}{0}$ , So  $f(5)$  *does not exist*.

**(B) (Finding Limits.)** Find  $\lim_{x \rightarrow c} f(x)$  for  $c = 1, 3, 5$ .

**Solution:**

**Computing the limit as  $x \rightarrow 1$**

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{(x-1)(x-5)}{(x-3)(x-5)} \stackrel{\text{Theorem 3}}{=} \frac{((1)-1)((1)-5)}{((1)-3)((1)-5)} = \frac{(0)(-4)}{(-2)(-4)} = \frac{0}{8} = 0.$$

limit of rational function with  $x=1$  in its domain

can substitute  $x=1$

Note that we are taking the limit of a rational function as  $x \rightarrow 1$ . Since  $x = 1$  is in the domain of the rational function, *Theorem 3* allows us to simply substitute in  $x = 1$ .

## Computing the limit as $x \rightarrow 3$

We first investigate what happens if we try substituting  $x = 3$  into the expression

$$\frac{(x-1)(x-5)}{(x-3)(x-5)} \xrightarrow{\text{try substituting } x=3} \frac{((3)-1)((3)-5)}{((3)-3)((3)-5)} = \frac{(2)(-2)}{(0)(-2)} = \frac{-4}{0}$$

Because we reach the expression  $\frac{-4}{0}$ , we know that **we should not substitute  $x = 3$ !**

(Remember that none of the Limit Properties allow us to substitute an  $x$  value into an expression if it results in a denominator of 0.)

But what are we to make of the fact that when we substitute  $x = 3$  into the expression, we reach the expression  $\frac{-4}{0}$ ??

Observe that simply substituting  $x = 3$  into the expression really amounts to taking the limit of the numerator and the limit of the denominator separately. To help you understand what that means, I'll do those limits more explicitly on the next page.

Take the limits of the numerator and denominator separately.

- Limit of numerator is  $\lim_{x \rightarrow 3} \underbrace{(x - 1)(x - 5)}_{\text{limit of a polynomial}} \stackrel{\text{Thm 3}}{=} \underbrace{((3) - 1)((3) - 5)}_{\text{can substitute } x=3} = (2)(-2) = -4.$
- Limit of denominator is  $\lim_{x \rightarrow 3} \underbrace{(x - 3)(x - 5)}_{\text{limit of a polynomial}} \stackrel{\text{Thm 3}}{=} \underbrace{((3) - 3)((3) - 5)}_{\text{can substitute } x=3} = (0)(-2) = 0.$

Since the limit of the numerator is *non-zero* while the limit of the denominator is *zero*, *Theorem 4* tells us that the limit *does not exist*.

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{(x - 1)(x - 5)}{(x - 3)(x - 5)} \stackrel{\text{Theorem 4}}{=} \text{DNE}.$$

## Computing the limit as $x \rightarrow 5$

We first investigate what happens if we try substituting  $x = 5$  into the expression

$$\frac{(x - 1)(x - 5)}{(x - 3)(x - 5)} \xrightarrow{\text{try substituting } x=5} \frac{((5) - 1)((5) - 5)}{((5) - 3)((5) - 5)} = \frac{0}{0}$$

Because we reach the expression  $\frac{0}{0}$ , we know that **we should not substitute  $x = 5$ !** (Again, remember that none of the Limit Properties allow us to substitute an  $x$  value into an expression if it results in a denominator of 0.)

But what are we to make of the fact that when we substitute  $x = 5$  into the expression, we reach the expression  $\frac{0}{0}$ ??

Observe that simply substituting  $x = 5$  into the expression really amounts to taking the limit of the numerator and the limit of the denominator separately. To help you understand what that means, I'll do those limits more explicitly on the next page.

Take the limits of the numerator and denominator separately.

- Limit of numerator is  $\lim_{x \rightarrow 5} \underbrace{(x - 1)(x - 5)}_{\text{limit of a polynomial}} \stackrel{\text{Thm 3}}{=} \underbrace{((5) - 1)((5) - 5)}_{\text{can substitute } x=5} = (4)(0) = 0.$

- Limit of denominator is  $\lim_{x \rightarrow 5} \underbrace{(x - 3)(x - 5)}_{\text{limit of a polynomial}} \stackrel{\text{Thm 3}}{=} \underbrace{((5) - 3)((5) - 5)}_{\text{can substitute } x=5} = (2)(0) = 0.$

Since the limits of the numerator and denominator by themselves are both zero, we realize that the limit is an *indeterminate form*. Recall that definition.

### **DEFINITION Indeterminate Form**

If  $\lim_{x \rightarrow c} f(x) = 0$  and  $\lim_{x \rightarrow c} g(x) = 0$ , then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form**.



Remember that when one encounters a limit that is an *indeterminate form*,

*“...Some more steps need to be done in order to convert the limit into one of the forms for which our theorems about limits (Theorems 2,3,4) can be used. Only then can a determination be made about the existence and value of the limit....”.*

The steps involve *cancelling terms* in the numerator and denominator. Remember that cancelling terms involves the most important concept of the first month of calculus:

***When can one cancel terms, and why?***

I'll do the limit in detail on the next page.

Computing the limit as  $x \rightarrow 5$ .

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} \frac{(x-1)(x-5)}{(x-3)(x-5)} \underset{\substack{\text{can cancel} \\ \text{because} \\ x-5 \neq 0}}{=} \lim_{x \rightarrow 5} \frac{(x-1)}{(x-3)} \underset{\substack{\text{Theorem 3} \\ \text{limit of rational} \\ \text{function with} \\ x=5 \text{ in its domain} \\ \text{(not indeterminate)}}}{=} \frac{((5)-1)}{((5)-3)} \underset{\text{can substitute } x=5}{=} \frac{(4)}{(2)} = 2.$$

More detailed justification for the Cancellation Step is as follows:

Since  $x \rightarrow 5$ , we know that  $x$  is getting closer and closer to 5 but not equal to 5.

So we know that  $x \neq 5$  and, therefore,  $x - 5 \neq 0$ .

So we know that we can cancel  $\frac{(x-5)}{(x-5)}$ .

Observe that the limit proceeded in the manner that was described a earlier in the video:

*We will see that when we encounter a limit that is an indeterminate form, we will be doing steps involving cancelling terms, in order to convert the limit to a new form that is not an indeterminate form. Then, we will be able to apply one of our theorems about limits (Theorems 2,3,4) to determine if the limit exists and, if it does exist, to give its value.*

## (C) Confirm Results with a Graph

Let's summarize our results of (A) and (B), and consider what they tell us about the graph of

$$f(x) = \frac{x^2 - 6x + 5}{x^2 - 8x + 15} = \frac{(x - 1)(x - 5)}{(x - 3)(x - 5)}$$

**At  $x = 1$**

- In (A), we found that  $f(1) = 0$ .

This tells us that there is a point on the graph with coordinates  $(x, y) = (1, 0)$ .

This is an  $x$  intercept caused by the factor of  $(x - 1)$  in the numerator.

- In (B), we found that  $\lim_{x \rightarrow 1} f(x) = 0$ .

This tells us that the graph is heading for the location  $(x, y) = (1, 0)$ .

## At $x = 3$

- In (A), we found that  $f(3)$  does not exist.

This tells us that there is no point on the graph with  $x$  coordinate  $x = 3$ .

- In (B), we found that  $\lim_{x \rightarrow 3} f(x)$  does not exist.

This tells us that as  $x$  approaches 3, there is no  $(x, y)$  location that the graph is heading towards.

We don't know yet how this will look. But the behavior is caused by the factor  $\frac{1}{(x-3)}$ .

## At $x = 5$

- In (A), we found that  $f(5)$  does not exist.

This tells us that there is no point on the graph with  $x$  coordinate  $x = 5$ .

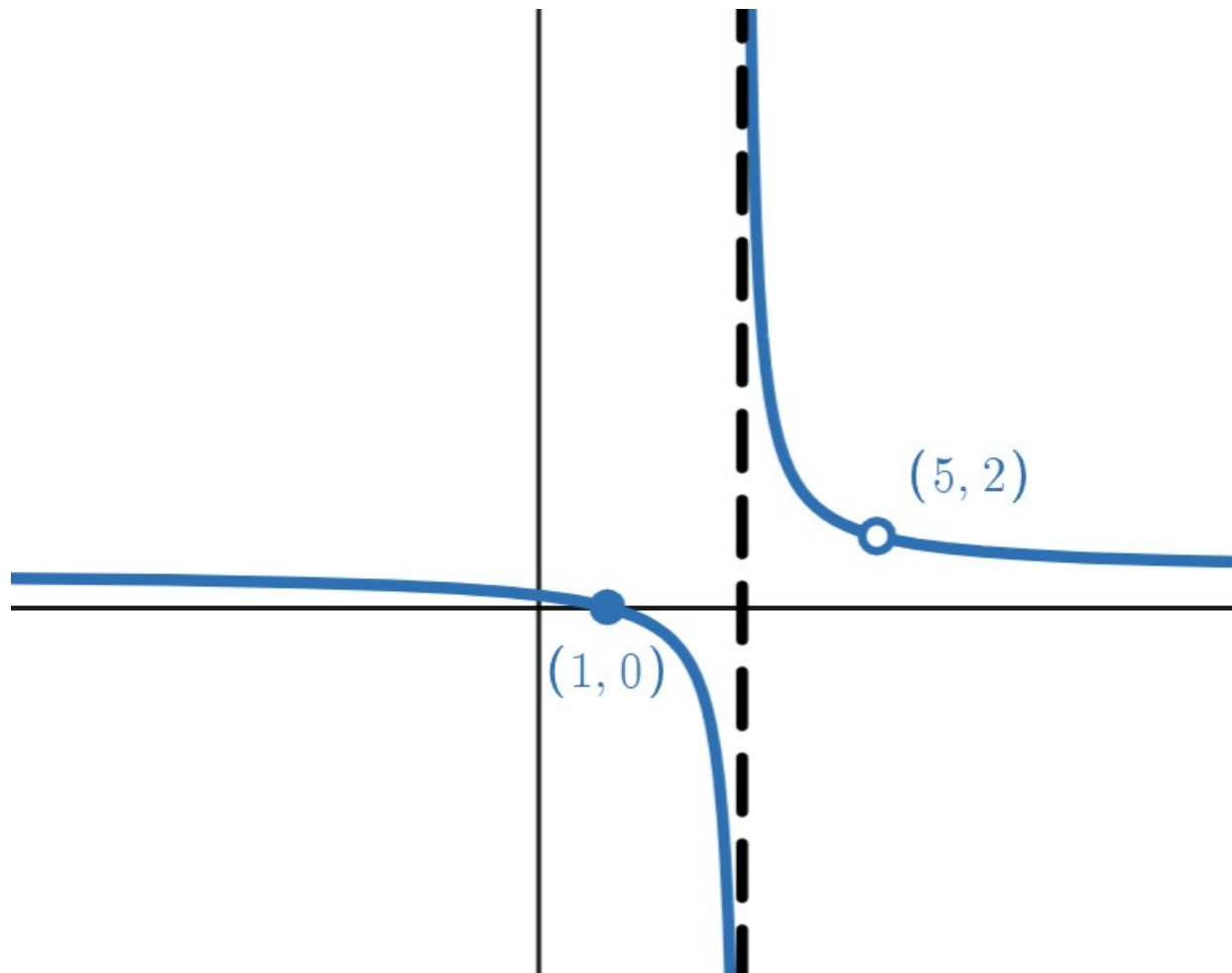
- In (B), we found that  $\lim_{x \rightarrow 5} f(x) = 2$ .

This tells us that the graph is heading for the location  $(x, y) = (5, 2)$ .

- Since the graph is heading for the location  $(x, y) = (5, 2)$ , but there is no point on the graph with  $x$  coordinate  $x = 5$ , we realize there is a *hole in the graph* at the location  $(x, y) = (5, 2)$ . The hole is caused by the factors  $\frac{(x-5)}{(x-5)}$  in the formula for  $f(x)$ .

We can observe the features that we have described in a graph of  $f(x)$ .

$$f(x) = \frac{x^2 - 6x + 5}{x^2 - 8x + 15} = \frac{(x - 1)(x - 5)}{(x - 3)(x - 5)}$$



## Observations:

- There is an  $x$  intercept at the location  $(x, y) = (1, 0)$ , caused by the factor of  $(x - 1)$  in the numerator. In our analysis, we found that  $f(1) = 0$  and  $\lim_{x \rightarrow 1} f(x) = 0$ .
- There is a *hole in the graph* at the location  $(x, y) = (5, 2)$ . The hole is caused by the factors  $\frac{(x-5)}{(x-5)}$  in the formula for  $f(x)$ . In our analysis, we found that  $f(5)$  *DNE* but  $\lim_{x \rightarrow 5} f(x) = 2$ .
- As  $x \rightarrow 3$ , there is no single  $(x, y)$  location that the graph is heading towards. This agrees with our analysis that showed that the  $\lim_{x \rightarrow 3} f(x)$  does not exist. (The graph has a *vertical asymptote* at  $x = 3$ . We will revisit this example in a future video, when we discuss an *Expanded Definition of Limit*.)

**End of [Example 2]**

**End of Video**