

# Limits Video E: Limits of Difference Quotients

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## Topics in this Video

- Example of computing the limit of a *difference quotient*.
- The importance of using Proper Limit Notation
- Why are *difference quotients* called that?
- Another example of computing the limit of a *difference quotient*.

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## Useful Definitions and Properties of Limits from *Limits Videos A and B*

### The Definition of *Limit*

**Symbol:**  $\lim_{x \rightarrow c} f(x) = L.$

**Spoken:** “The limit, as  $x$  approaches  $c$ , of  $f(x)$  is  $L$ .”

**Less-Abbreviated Symbol:**  $f(x) \rightarrow L$  as  $x \rightarrow c$ .

**Spoken:** “ $f(x)$  approaches  $L$  as  $x$  approaches  $c$ .”

**Usage:**  $x$  is a variable,  $f$  is a function,  $c$  is a real number, and  $L$  is a real number.

**Meaning:** as  $x$  gets closer and closer to  $c$ , but not equal to  $c$ , the value of  $f(x)$  gets closer and closer to  $L$  (may actually equal  $L$ ).

**Graphical Significance:** The graph of  $f$  appears to be heading for location  $(x, y) = (c, L)$  from both sides.

## THEOREM 2 Properties of Limits

Let  $f$  and  $g$  be two functions, and assume that

$$\lim_{x \rightarrow c} f(x) = L \quad \lim_{x \rightarrow c} g(x) = M$$

where  $L$  and  $M$  are real numbers (both limits exist). Then

1.  $\lim_{x \rightarrow c} k = k$  for any constant  $k$
2.  $\lim_{x \rightarrow c} x = c$
3.  $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = L + M$
4.  $\lim_{x \rightarrow c} [f(x) - g(x)] = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x) = L - M$
5.  $\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x) = kL$  for any constant  $k$
6.  $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = [\lim_{x \rightarrow c} f(x)][\lim_{x \rightarrow c} g(x)] = LM$
7.  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{L}{M}$  if  $M \neq 0$
8.  $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)} = \sqrt[n]{L}$  if  $L > 0$  or  $n$  is odd

### **THEOREM 3** Limits of Polynomial and Rational Functions

1.  $\lim_{x \rightarrow c} f(x) = f(c)$  for  $f$  any polynomial function.
2.  $\lim_{x \rightarrow c} r(x) = r(c)$  for  $r$  any rational function with a nonzero denominator at  $x = c$ .

### **DEFINITION** Indeterminate Form

If  $\lim_{x \rightarrow c} f(x) = 0$  and  $\lim_{x \rightarrow c} g(x) = 0$ , then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form**.

## Most Important Concept of the First Month of Calculus

A key concept in this video is the most important concept of the first month of Calculus:

*When can one cancel terms, and why?*

Observe that

- We can cancel a quotient of like terms if they are non-zero. For example,  $\frac{5}{5} = 1$ .
- But we cannot cancel  $\frac{0}{0}$ . The expression  $\frac{0}{0}$  does not equal 1. (It also does not equal 0.) The expression  $\frac{0}{0}$  does not define a real number. We say that  $\frac{0}{0}$  is *undefined*, or that it *does not exist (DNE)*.
- Without knowing anything about the value of the variable  $x$ , we cannot cancel  $\frac{x}{x}$ . (If it turns out that  $x \neq 0$ , then  $\frac{x}{x} = 1$ . But if it turns out that  $x = 0$ , then  $\frac{x}{x}$  is *undefined*. So we can't say in advance.)
- But if we are given extra information enables us to determine that  $x \neq 0$  or  $x = 0$ , then we can replace the expression  $\frac{x}{x}$  with either a 1 or the expression *DNE*.

## First Example of computing the limit of a *difference quotient*.

**[Example 1]** For  $f(x) = x^2 - 6x + 5$ , find the following limit.

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h}$$

**Remark:** This is an extremely important type of limit. The expression inside the limit is called a *difference quotient*. I will talk about why the expression is given that name later. For now, we will focus on just finding the limit.

### **Solution:**

We will follow this **strategy** to find  $\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h}$ .

- Build the parts represented by the symbols in the difference quotient.
- Put those parts into the limit expression
- Find the limit. One of the steps will involve cancelling.

**Build the parts represented by the symbols in the difference quotient.**

We need two things in order to build  $\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h}$

- We need  $f(4)$
- We need  $f(4 + h)$

For both of these things, it is useful to have the *empty version* of the function  $f$ .

The function  $f$  with a variable inside is

$$f(x) = x^2 - 6x + 5$$

The *empty version* is

$$f(\quad) = (\quad)^2 - 6(\quad) + 5 \quad \textit{empty version}$$

To find  $f(4)$ , we put 4 in the parentheses and simplify

$$\begin{aligned} f(4) &= (4)^2 - 6(4) + 5 \\ &= 16 - 24 + 5 \\ &= -3 \end{aligned}$$



To find  $f(4 + h)$ , we put  $4 + h$  in the parentheses and simplify

$$\begin{aligned} f(4 + h) &= (4 + h)^2 - 6(4 + h) + 5 \\ &= 16 + \underline{8h} + h^2 - 24 - 6h + 5 \\ &= h^2 + 2h - 3 \end{aligned}$$

### **Beware of a common mistake!**

When computing  $(4 + h)^2$ , students will often write

$$(4 + h)^2 = 4^2 + h^2 = 16 + h^2$$

But this is incorrect!

The correct computation results in the “cross term”  $8h$  shown above and in the line below.

$$(4 + h)^2 = (4 + h)(4 + h) = 4h + 4h + 4h + h^2 = 16 + \underline{8h} + h^2$$

**Put the parts into the limit expression.**

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \text{ put in parts } = \lim_{h \rightarrow 0} \frac{(h^2 + 2h - 3) - (-3)}{h} ..$$

## Find the limit.

We first investigate what happens if we try substituting  $h = 0$  into the expression

$$\frac{(h^2 + 2h - 3) - (-3)}{h} \xrightarrow{\text{try substituting } h=0} \frac{((0)^2 + 2(0) - 3) - (-3)}{(0)} = \frac{0}{0}$$

Because we reach the expression  $\frac{0}{0}$ , we know that **we should not substitute  $h = 0$ !**

(Remember that none of the Limit Properties allow us to substitute an value for the variable into an expression if it results in a denominator of 0.)

But what are we to make of the fact that when we substitute  $h = 0$  into the expression, we reach the expression  $\frac{0}{0}$ ??

Observe that simply substituting  $h = 0$  into the expression really amounts to taking the limit of the numerator and the limit of the denominator separately. To help you understand what that means, I'll do those limits more explicitly on the next page.

Take the limits of the numerator and denominator separately.

- Lim of numerator:  $\lim_{h \rightarrow 0} (h^2 + 2h - 3) - (-3) \stackrel{\text{Thm 3}}{=} \underbrace{((0)^2 + 2(0) - 3) - (-3)}_{\text{can substitute } h=0} = (4)(0) = 0.$   
limit of polynomial in variable  $h$

- Lim of denominator:  $\lim_{h \rightarrow 0} h \stackrel{\text{Thm 3}}{=} \underbrace{(0)}_{\text{can substitute } h=0} = 0.$   
limit of polynomial in variable  $h$

Since the limits of the numerator and denominator by themselves are both zero, we realize that the limit is an *indeterminate form*.

### **DEFINITION Indeterminate Form**

If  $\lim_{x \rightarrow c} f(x) = 0$  and  $\lim_{x \rightarrow c} g(x) = 0$ , then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form**.

Remember that when one encounters a limit that is an *indeterminate form*,

*“...Some more steps need to be done in order to convert the limit into one of the forms for which our theorems about limits (Theorems 2,3,4) can be used. Only then can a determination be made about the existence and value of the limit....”.*

The steps involve *cancelling terms* in the numerator and denominator. Remember that cancelling terms involves the most important concept of the first month of calculus:

***When can one cancel terms, and why?***

I'll do the limit in detail on the next page.

Computing the limit.

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{(h^2 + 2h - 3) - (-3)}{h}$$

indeterminate form

put in parts

$$= \lim_{h \rightarrow 0} \frac{h^2 + 2h}{h}$$

indeterminate form

simplify

$$= \lim_{h \rightarrow 0} \frac{h(h+2)}{h}$$

indeterminate form

factor

Since  $h \rightarrow 0$ , we know  $h \neq 0$ , so we can cancel  $\frac{h}{h}$ .

$$= \lim_{h \rightarrow 0} h + 2$$

limit of polynomial  
in the variable  $h$   
(no longer indeterminate)

$$= (0) + 2 = 2$$

Thm 3

**End of [Example 1]**

## Observations about [Example 1]:

- We started with an *indeterminate form*.
- We had two rows of simplifying and rewriting the expression inside the limit, until we got it to a form that was still indeterminate, but a form in which we could see clearly what was causing the limit to be indeterminate:

It was the  $\frac{h}{h}$  that was causing the limit to be indeterminate.

- Then we canceled  $\frac{h}{h}$  using the fact that we knew that  $h \neq 0$ . This was an important step (involving the most important concept of the first month of calculus!), so we explained it.
- Then, the limit was no longer an indeterminate form, and we could use Theorem 3.
- Observe that the limit symbol appears at the beginning of every expression in the first four rows of the table. In all of those expressions, we have not yet done the limit. It is not until the fifth row, when we use Theorem 3, that the limit symbol goes away.





## First invalid equal sign

- The expression  $\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h}$  represents a *number*. (We found in [Example 1] that it represents the number 2.)
- The expression  $\frac{(h^2 + 2h - 3) - (-3)}{h}$  is a *function* in the variable  $h$ .

A *number* is not the same thing as a *function* in the variable  $h$ , so the first equal sign is invalid!

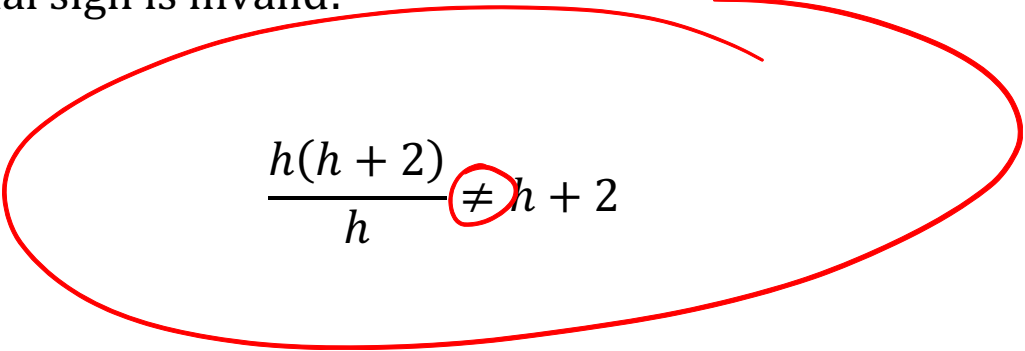
$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \neq \frac{(h^2 + 2h - 3) - (-3)}{h}$$

## Second invalid equal sign

- The expression  $\frac{h(h+2)}{h}$  is a *function* in the variable  $h$ , with domain all  $h \neq 0$ .
- The expression  $h + 2$  is a *function* in the variable  $h$ , with domain all real numbers  $h$ .

These functions are *not equal*, because they have *different domains*!

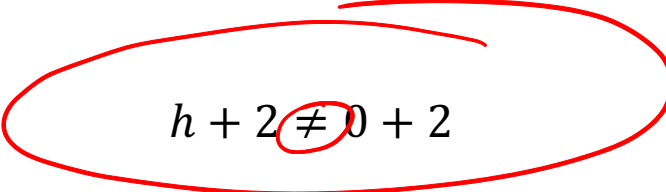
Therefore, the fourth equal sign is invalid!


$$\frac{h(h+2)}{h} \neq h+2$$

### Third invalid equal sign

- The expression  $h + 2$  is a *function* in the variable  $h$ .
- The expression  $0 + 2$  is a *number*.

A *function* in the variable  $h$  is not the same thing as a *number*, so the fifth equal sign is invalid!


$$h + 2 = 0 + 2$$

### Conclusion about limit notation.

A correction solution to a limit problem should show not only the correct final answer for the limit, but also correct steps, with correct limit notation and explanation of the steps (as modeled in my solution to **Example 1**).

## Why are *difference quotients* called that?

Having done an example, let's now discuss why the expression  $\frac{f(4+h)-f(4)}{h}$  is called a *difference quotient*.

Recall what the symbol  $f(4)$  denotes: the  $y$  value on the graph of  $f$  when  $x = 4$ .

So the symbol  $f(4 + h)$  denotes: the  $y$  value on the graph of  $f$  when  $x = 4 + h$ .

In other words, when  $h \neq 0$ , the symbols  $(4, f(4))$  and  $(4 + h, f(4 + h))$  are the  $(x, y)$  coordinates of two points on the graph of  $f$ .

So the expression  $f(4 + h) - f(4)$  represents a change in  $y$ . That is,  $\Delta y = f(4 + h) - f(4)$ .

The change in  $x$  would be  $\Delta x = (4 + h) - (4) = h$ .

Observe that the expression  $\frac{f(4+h)-f(4)}{h}$  is the result of computing the change in  $y$  divided by the change in  $x$ .

$$\frac{\Delta y}{\Delta x} = \frac{f(4+h) - f(4)}{(4+h) - (4)} = \frac{f(4+h) - f(4)}{h}.$$

Instead of saying *the change in  $y$  divided by the change in  $x$* .

we could say *the difference in  $y$  divided by the difference in  $x$* .

In other words, a *quotient of differences*. Hence, the name *difference quotient*.

The next example will be included in the printed notes accompanying this video, but I will not discuss the example in the video. (The concepts and solution strategy are the same as in **[Example 1]**.) You should study the printed notes carefully on your own.

**[Example 2]** For  $f(x) = 7x - 5$ , find the following limit.

$$\lim_{h \rightarrow 0} \frac{f(4 + h) - f(4)}{h}$$

**Solution:**

Observe that this is the limit of a *difference quotient*.

We will follow the same **strategy** that we used in **[Example 1]** to find the limit.

- Build the parts represented by the symbols in the difference quotient.
- Put those parts into the limit expression
- Find the limit. One of the steps will involve cancelling.

**Build the parts represented by the symbols in the difference quotient.**

The function  $f$  with a variable inside is  $f(x) = 7x - 5$

The *empty version* of  $f$  is  $f(\quad) = 7(\quad) - 5$

To find  $f(4)$ , put 4 in the parentheses and simplify

$$\begin{aligned}f(4) &= 7(4) - 5 \\ &= 28 - 5 \\ &= 23\end{aligned}$$

To find  $f(4 + h)$ , put  $4 + h$  in the parentheses and simplify

$$\begin{aligned}f(4 + h) &= 7(4 + h) - 5 \\ &= 28 + 7h - 5 \\ &= 23 + 7h\end{aligned}$$

**Put the parts into the limit expression.**

$$\lim_{h \rightarrow 0} \frac{f(4 + h) - f(4)}{h} \underset{\text{put in parts}}{=} \lim_{h \rightarrow 0} \frac{(23 + 7h) - (23)}{h}.$$



## Find the limit.

We first investigate what happens if we try substituting  $h = 0$  into the expression

$$\frac{(23 + 7h) - (23)}{h} \quad \xRightarrow{\text{try substituting } h=0} \quad \frac{(23 + 7(0)) - (23)}{(0)} = \frac{0}{0}$$

Because we reach the expression  $\frac{0}{0}$ , we know that **we should not substitute  $h = 0$ !**

(Remember that none of the Limit Properties allow us to substitute an value for the variable into an expression if it results in a denominator of 0.)

But what are we to make of the fact that when we substitute  $h = 0$  into the expression, we reach the expression  $\frac{0}{0}$ ??

Observe that simply substituting  $h = 0$  into the expression really amounts to taking the limit of the numerator and the limit of the denominator separately. To help you understand what that means, I'll do those limits more explicitly on the next page.

Take the limits of the numerator and denominator separately.

- Lim of numerator:  $\lim_{h \rightarrow 0} (23 + 7h) - (23) \stackrel{\text{Thm 3}}{=} \underbrace{(23 + 7(0)) - (23)}_{\text{can substitute } h=0} = 0.$   
limit of polynomial in variable  $h$

- Lim of denominator:  $\lim_{h \rightarrow 0} h \stackrel{\text{Thm 3}}{=} \underbrace{(0)}_{\text{can substitute } h=0} = 0.$   
limit of polynomial in variable  $h$

Since the limits of the numerator and denominator by themselves are both zero, we realize that the limit is an *indeterminate form*.

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The steps involve *cancelling terms* in the numerator and denominator. Remember that cancelling terms involves the most important concept of the first month of calculus:

***When can one cancel terms, and why?***

I'll do the limit in detail on the next page.

Computing the limit.

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \stackrel{\text{put in parts}}{=} \lim_{h \rightarrow 0} \underbrace{\frac{(23+7h) - (23)}{h}}_{\text{indeterminate form}}$$

$$\stackrel{\text{simplify}}{=} \lim_{h \rightarrow 0} \underbrace{\frac{7h}{h}}_{\text{indeterminate form}}$$

Since  $h \rightarrow 0$ , we know  $h \neq 0$ , so we can cancel  $\frac{h}{h}$ .

$$\stackrel{\text{cancel}}{=} \lim_{h \rightarrow 0} \underbrace{7}_{\substack{\text{limit of a constant} \\ \text{(no longer indeterminate)}}$$

$$\stackrel{\text{Thm 2.1}}{=} 7$$

**End of [Example 1]**

## Observations about [Example 2]:

- We started with an indeterminate form
- We had one row of simplifying the limit, putting it into a form that was still indeterminate, but a form in which we could see clearly what was causing the limit to be indeterminate:

It was the  $\frac{h}{h}$  that was causing the limit to be indeterminate.

- Then we canceled  $\frac{h}{h}$  using the fact that we knew that  $h \neq 0$ . This was an important step (involving the most important concept of the first month of calculus!), so we explained it.
- Then, the limit was no longer an indeterminate form, and we could use Theorem 2.1.
- Observe that the limit symbol appears at the beginning of every expression in the first three rows of the table. In all of those expressions, we have not yet done the limit. It is not until the fourth row, when we use Theorem 2.1, that the limit symbol goes away.

**End of Video**