In this video, we will take an *analytical* approach to Infinite Limits and Vertical Asymptotes. That is, function f is given by a formula, not graph.

Recall the limit notation that we have discussed in previous videos, and the trends that the notation represents.

		Trend in the y-values			
		y values approach the number L (and may equal L)	y values grow negative, without bound	y values grow positive, without bound	y values do anything else
Trend in the x-values	x approaches the number c from the left	$\lim\limits_{x\to c^-} f(x) = L$ (section 2.1)	$\lim\limits_{x\to c^-} f(x) = -\infty$ (section 2.2)	$\lim\limits_{x\to c^-} f(x) = \infty$ (section 2.2)	$\lim\limits_{x\to c^-} f(x)\ DNE$
	x approaches the number c from right	$\lim\limits_{x\to c^+} f(x) = L$ (section 2.1)	$\lim\limits_{x\to c^+} f(x) = -\infty$ (section 2.2)	$\lim\limits_{x\to c^+} f(x) = \infty$ (section 2.2)	$\lim\limits_{x\to c^+} f(x)\ DNE$
	x approaches the number c	$\lim\limits_{x\to c} f(x) = L$ (section 2.1)	$\lim\limits_{x\to c} f(x) = -\infty$ (section 2.2)	$\lim\limits_{x\to c} f(x) = \infty$ (section 2.2)	$\lim\limits_{x\to c} f(x)\ DNE$
	x grows negative, without bound	$\lim\limits_{x\to -\infty} f(x) = L$ (section 2.2)	$\lim\limits_{x\to -\infty} f(x) = -\infty$ (section 2.2)	$\lim\limits_{x\to -\infty} f(x) = \infty$ (section 2.2)	$\lim\limits_{x\to -\infty} f(x)\ DNE$
	x grows positive, without bound	$\lim\limits_{x\to \infty} f(x) = L$ (section 2.2)	$\lim\limits_{x\to \infty} f(x) = -\infty$ (section 2.2)	$\lim\limits_{x\to \infty} f(x) = \infty$ (section 2.2)	$\lim\limits_{x\to \infty} f(x)\ DNE$
	x values do anything else	no terminology in Section 2.1 or 2.2	no terminology in Section 2.1 or 2.2	no terminology in Section 2.1 or 2.2	no terminology in Section 2.1 or 2.2

[Example 1] For the function

$$f(x) = \frac{5}{x-7}$$

find the function value and limits listed below and explain what they tell us about the graph of $f(x)$. Use the terminology and notation of infinity, where applicable. (Concepts from Section 2.2)

(A) $f(7)$

(B) $\lim_{x \to 7^-} f(x)$

(C) $\lim_{x \to 7^-} f(x)$

(D) $\lim_{x \to 7^-} f(x)$

Solution to (A)

$$f(7) = \frac{5}{(7)-7} = \frac{5}{0} \text{ does not exist}$$

This tells us that there is no point on the graph with $X=7$.

Solution to (B)

We are being asked to compute $\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} \frac{5}{x-7}$

First, recall how we would have done this using Section 2.1 techniques

Observe that the limit of the numerator is

$$\lim_{x \to 7^-} \text{numerator} = \lim_{x \to 7^-} 5 \overset{\text{Theorem 2.1}}{=} 5$$

And the limit of the denominator is

$$\lim_{x \to 7^-} \text{denominator} = \lim_{x \to 7^-} \underbrace{x - 7}_{\text{Polynomial}} \underset{\text{use Theorem 3}}{=} (7) - 7 = 0$$

So the limit of numerator is not zero, and the limit of the denominator is zero.

Therefore, we know that

$\lim_{x \to 7^-} \frac{5}{x-7}$ does not exist, by theorem 4 of Section 2.1

This tells us that there is no limit in the sense of Section 2.1. That is, there is no real number L that the y values are getting closer and closer to.

But we are told to use the terminology and notation of *infinity*, where applicable. (Concepts from Section 2.2)

How would this limit be done using Section 2.2 techniques?

Remember the difference between Section 2.1 and Section 2.2 limit terminology and notation. In Section 2.2, we have terminology and notation for abbreviating the descriptions of more kinds of trends in the x and y values of a function. To use Section 2.2 terminology, we will need to have information about what the x and y values are doing, in order to see if we can discern a trend.

One way is to make a table of x, y values.

How should we populate it?

We are being asked to find $\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} \frac{5}{x-7}$. Part of this symbol is $x \to 7^-$. This tells us that we need to investigate x values that have the following *trend*:

x is getting closer and closer to 7, but <u>less than 7</u>.

So we build a column of x values that are doing that, and then compute the corresponding y values for the second column.

x	$y = \frac{5}{x-7}$
6.9	$y = \frac{5}{(6.9)-7} = \frac{5}{-0.1} = -50$
6.99	$y = \frac{5}{(6.99)-7} = \frac{5}{-0.01} = -500$
6.999	$y = \frac{5}{(6.999)-7} = \frac{5}{-0.001} = -5000$

Observe that we see a trend in the resulting y values:

The y values are getting more and more <u>negative</u>, without bound.

For clarity, lets write a single sentence description that describes both the trend in the x values and the trend in the y values.

As x gets closer and closer to 7, but less than 7, the values of f(x) get more and more negative, without bound.

We recognize that this sentence description has the following abbreviation in limit notation.

$$\lim_{x \to 7^-} f(x) = -\infty$$

Which is spoken

The limit, as x approaches 7 from the left, of f(x) is negative infinity.

This tells us the following about how the graph behaves:

The graph has a vertical asymptote with line equation $x=7$, and the graph goes down along the left side of that asymptote.

Remark new terminology:

Remember what I mentioned in a previous video about this situation. In the Barnett book, in Section 2.2, the authors would write the symbol $\lim_{x\to 7^-} \frac{5}{x-7} = -\infty$, but they would say that *the limit does not exist*. The reason they would do that is because using Section 2.1 techniques (Theorem 4), there is no limit in the sense of Section 2.1. That is, there is no real number L that the y values are getting closer and closer to.

I don't like saying that the limit does not exist when we know that $\lim_{x\to 7^-} f(x) = -\infty$. Saying that the limit does not exist obscures what we have learned about the function, and it does not convey that we have expanded our definition of what a limit can be.

So in my videos, if I find that

$$\lim_{x\to c} f(x) = \infty$$

then I will say that

The limit, as x approaches c, of $f(x)$ is infinity.

Solution to (C)

We are being asked to compute $\lim_{x\to 7^+} f(x) = \lim_{x\to 7^+} \frac{5}{x-7}$

Remark:

We can quickly observe what would happen if we were to use Section 2.1 techniques:

$\lim_{x\to 7^+} f(x) = \lim_{x\to 7^+} \frac{5}{x-7}$ Does Not Exist, by Theorem 4 of Section 2.1

observe that the limit of the numerator is 5

observe that the limit of the denominator is 0

But we are asked to use the terminology and notation of infinity, where applicable. That is, we are to use techniques of Section ~~2.1~~ 2.2

So we will make a table of x, y values.

In the current question (C), we have $x \to 7^+$. This tells us that we need to investigate x values that have the following *trend*:

X is getting closer and closer to 7, but greater than 7.

So we build a column of x values that are doing that, and then compute the corresponding y values for the second column.

x	$y = \frac{5}{x-7}$
7.1	$\frac{5}{(7.1)-7} = \frac{5}{0.1} = 50$
7.01	$\frac{5}{(7.01)-7} = \frac{5}{0.01} = 500$
7.001	$\frac{5}{(7.001)-7} = \frac{5}{.001} = 5000$

Observe that we see a trend in the resulting y values. Here is the sentence summary

When x gets closer and closer to 7, but greater than 7,
the y values get more and more positive, without bound.

The corresponding abbreviation in limit notation is

$$\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} \frac{5}{x-7} = \infty$$

Which is spoken

The limit, as x approaches 7 from the right, of f(x) is infinity

The corresponding graph behavior is.

The graph has a vertical asymptote with line equation x = 7, and the graph goes up along the right side of the asymptote.

Solution to (D)

We are asked to find the two-sided limit. That is easy:

The $\lim_{x \to 7} f(x) = \lim_{x \to 7} \frac{5}{x-7}$ does not exist, because the left and right limits don't match

Remark on Invalid Solutions:

In problems about limits involving infinity, there are some common invalid solution methods. Most of the time, these solution methods give the incorrect answer. Sometimes they happen to give the correct answer. But regardless of whether or not they happen to give the correct answer, the solutions are invalid. I will present two of the common invalid solutions to questions (B),(C),(D) that we did above. Note that we did not solve (B),(C),(D) this way when we solved **[Example 1]**.

We did not do this:

Common Invalid solution

(B) left limit: $\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} \frac{5}{x-7} = \frac{5}{7-7} = \frac{5}{0}$ DNE

(C) right limit: $\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} \frac{5}{x-7} = \frac{5}{7-7} = \frac{5}{0}$ DNE

(D) limit: $\underbrace{\lim_{x \to 7} f(x) = \lim_{x \to 7} \frac{5}{x-7} = \frac{5}{7-7} = \frac{5}{0}}_{\text{but invalid solution method.}}$ DNE ✓ correct answer

invalid solutions

And we also did not do this:

Another common invalid solution

(B) left limit: $\lim\limits_{x \to 7^-} f(x) = \lim\limits_{x \to 7^-} \frac{5}{x-7} = \frac{5}{(7)-7} = \frac{5}{0} = \infty$

(C) right limit: $\lim\limits_{x \to 7^+} f(x) = \lim\limits_{x \to 7^+} \frac{5}{x-7} = \frac{5}{(7)-7} = \frac{5}{0} = \infty$ correct answer

(D) limit: $\lim\limits_{x \to 7} f(x) = \lim\limits_{x \to 7} \frac{5}{x-7} = \frac{5}{(7)-7} = \frac{5}{0} = \infty$

invalid solutions.

For our second example, we will revisit a function that we studied in an earlier video about limit of rational functions.

[Example 2] For the function

$$f(x) = \underbrace{\frac{x^2 - 6x + 5}{x^2 - 8x + 15}}_{\text{standard form}} = \underbrace{\frac{(x-1)(x-5)}{(x-3)(x-5)}}_{\text{factored form}}$$

find the function value and limits listed below and explain what they tell us about the graph of $f(x)$. Use the terminology and notation of infinity, where applicable. (Concepts from Section 2.2)

(A) $f(3)$

(B) $\lim_{x \to 3^-} f(x)$

(C) $\lim_{x \to 3^-} f(x)$

(D) $\lim_{x \to 3^-} f(x)$

Solution to (A) use the factored form

$$f(3) = \frac{((3)-1)((3)-5)}{((3)-3)((3)-5)} = \frac{(2)(-2)}{(0)(-2)} = \frac{-4}{0} \quad \text{Does not exist}$$

There is no point on the graph with $x=3$

Solution to (B)

We are being asked for $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{(x-1)(x-5)}{(x-3)(x-5)}$

First, recall how we did this limit in an earlier video, using Section 2.1 techniques

We observed that the limit of the numerator is

$$\lim_{x \to 3^-} \text{numerator} = \lim_{x \to 3^-} \underbrace{(x-1)(x-5)}_{\text{Polynomial}} \underset{\text{use Theorem 3}}{=} ((3)-1)((3)-5) = (2)(-2) = -4$$

And the limit of the denominator is

$$\lim_{x \to 3^-} \text{denominator} = \lim_{x \to 3^-} \underbrace{(x-3)(x-5)}_{\text{Polynomial function}} \underset{\text{use theorem 3}}{=} ((3)-3)((3)-5) = (0)(-2) = 0$$

So the limit of numerator is not zero, and the limit of the denominator is zero.

Therefore, we know that

The limit of the ratio does not exist.

That is, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{\text{numerator}}{\text{denominator}}$ does not exist by theorem 4

This tolds us that there was no limit in the sense of Section 2.1. That is, there is no real number L that the y values are getting closer and closer to.

But in the current example, we are told to use the terminology and notation of *infinity*, where applicable. (Concepts from Section 2.2)

We can start by simplifying the limit by doing some cancelling. But we must do it carefully and explain why we can do it.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{(x-1)(x-5)}{(x-3)(x-5)} = \lim_{x\to 3^-} \frac{x-1}{x-3}$$

Since $x \to 3^-$
we know $x \neq 5$
So $x-5 \neq 0$
So we can cancel $\frac{x-5}{x-5}$

So our job has been simplified to finding $\lim_{x\to 3^-} \frac{(x-1)}{(x-3)}$. Part of this symbol is $x \to 3^-$. This tells us that we need to investigate x values that have the following *trend*:

x is getting closer and closer to 3, but less than 3.

So we build a column of x values that are doing that, and then compute the corresponding y values for the second column.

x	$y = \frac{x-1}{x-3}$
2.9	$\frac{(2.9)-1}{(2.9)-3} = \frac{1.9}{-0.1} = -19$
2.99	$\frac{(2.99)-1}{(2.99)-3} = \frac{1.99}{-0.01} = -199$
2.999	$\frac{(2.999)-1}{(2.999)-3} = \frac{1.999}{-0.001} = -1999$

Observe that we see a trend in the resulting y values. Here is the sentence summary

As x approaches 3 from the left, the y values get more & more negative, without bound.

We recognize that this sentence description has the following abbreviation in limit notation.

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x-1}{x-3} = -\infty$$

Which is spoken

The limit, as x approaches 3 from the left, of f(x) is negative infinity.

This tells us the following about how the graph behaves:

The graph of f(x) has a vertical asymptote with line equation $x = 3$, and the graph is going down along the left side of the asymptote.

Solution to (C)

We again start by simplifying the limit by doing some cancelling and explaining why we can do it.

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{(x-1)(x-5)}{(x-3)(x-5)} = \lim_{x\to 3^+} \frac{x-1}{x-3}$$

since $x \to 3^+$, we know that $x \neq 5$, so $x-5 \neq 0$, so we can cancel $\frac{x-5}{x-5}$

We have $x \to 3^+$. This tells us that we need to investigate x values that have the following *trend*:

X must be getting closer and closer to 3, but greater than 3.

So we build a column of x values that are doing that, and then compute the corresponding y values for the second column.

x	$y = \frac{x-1}{x-3}$
3.1	$\frac{(3.1)-1}{(3.1)-3} = \frac{2.1}{0.1} = 21$
3.01	$\frac{(3.01)-1}{(3.01)-3} = \frac{2.01}{0.01} = 201$
3.001	$\frac{(3.001)-1}{(3.001)-3} = \frac{2.001}{0.001} = 2001$

Observe that we see a trend in the resulting y values. Here is the sentence summary

As x approaches 3 from the right, the y values get more & more positive, without bound.

The corresponding abbreviation in limit notation is $\lim_{x \to 3^+} f(x) = \infty$.

Which is spoken The limit, as x approaches 3 from the right, of f(x) is infinity.

The corresponding graph behavior is.
The graph of f(x) has a vertical asymptote with line equation x=3, and the graph of f(x) goes up along the right side of that asymptote.

Solution to (D)

In question (D), we are asked to find the two-sided limit. That is easy:

$\lim_{x \to 3} f(x)$ does not exist because the left and right limits don't match.

[End of Example 2]

Remark:

Let's compare the limits of $f(x) = \frac{x^2-6x+5}{x^2-8x+15} = \frac{(x-1)(x-5)}{(x-3)(x-5)}$ when using Section 2.1 techniques the the results when using Section 2.2 techniques.

Using Section 2.1 Techniques

$\lim_{x \to 3^-} f(x)$ does not exist, by Theorem 4 (of section 2.1)

$\lim_{x \to 3^+} f(x)$ does not exist, by Theorem 4

$\lim_{x \to 3} f(x)$ does not exist, by Theorem 4.

Using Section 2.2 Techniques (the current example)

$$\lim_{x \to 3^-} f(x) = -\infty$$

$$\lim_{x \to 3^+} f(x) = \infty$$

$\lim_{x \to 3} f(x)$ does not exist, because the left and right limits don't match.

Not because of theorem 4.

End of Video