In this video, we will discuss the *end behavior* of *polynomial functions*.

Review of Polynomial Functions

Remember that a ***polynomial function*** is a function such as

$$f(x) = 6x^5 - 4x^3 + 11x - 13$$

or

$$f(x) = 6x^5 - 4x^8 + 11x$$

A polynomial consists of a sum of one or more expressions of the form cx^p, where c is a real number constant, called a ***coefficient***, and p is a *non-negative integer*. (That is, p is an integer and $p \geq 0$.)

The ***terms*** of a polynomial are the expressions of the form cx^p

That is, a term of a polynomial consists of a *power function* with non-negative integer power, multiplied by a real number *coefficient*.

Consider some special cases of polynomial functions.

Functions such as $f(x) = -3x + 5$. Notice that in equation form, this would be

$y = -3x + 5$

an equation for a line.

Functions such as $f(x) = 5$. This is a constant function. In equation form, this would be

$y = 5$

an equation for a horizontal line. This qualifies as a polynomial, because it can be written $f(x) = 5 = 5 \cdot 1 = 5x^0$. These kinds of polynomials would be called *constant polynomials*.

The function $f(x) = 0$. This is a particular constant function. In equation form, this would be

$y = 0$

an equation for a horizontal line. This qualifies as a polynomial, because it can be written $f(x) = 0 = 0 \cdot 1 = 0x^0$. It is called the *zero polynomial*. It is a special case of a *constant polynomial*.

The fact that the zero polynomial, $f(x) = 0$, qualifies to be called a polynomial is a bit of a nuisance. It makes definition of some of the terminology associated with polynomials a bit more complicated.

We can define the *leading term*, *degree*, *leading coefficient*, and *constant term* only for *non-zero* polynomials. They are not defined for the zero polynomial.

- The ***leading*** term of a non-zero polynomial is the non-zero term that has the highest power.
- The ***degree*** of a non-zero polynomial is the power appearing in its leading term.
- The ***leading coefficient*** of a non-zero polynomial is the coefficient of its leading term.
- The ***constant term*** of a non-zero polynomial is the term of the form $+c$, where c is a real number constant. The constant term can be zero.

These definitions can be better understood if we consider some examples.

[Example 1] For the function $f(x) = 6x^5 - 4x^3 + 11x - 13$

- the leading term is
- the degree is
- the leading coefficient is
- the constant term is

[Example 2] For the function $f(x) = 6x^5 - 4x^8 + 11x$

- the leading term is
- the degree is
- the leading coefficient is
- the constant term is

[Example 3] For the function $f(x) = -3x + 5$

- the leading term is $-3x$
- the degree is 1 because $x = x^1$
- the leading coefficient is -3
- the constant term is 5

[Example 4] For the function $f(x) = 5$ constant function

- the leading term is 5
- the degree is (rewrite $5 = 5 \cdot 1 = 5 \cdot x^0$) so the degree is 0
- the leading coefficient is 5
- the constant term is 5

[Example 5] For the function $f(x) = 0$ constant function, the zero polynomial

- the leading term is
- the degree is
- the leading coefficient is
- the constant term is

$\Big\}$ not defined for the zero polynomial

Observations

The leading term is not necessarily the one that appears at the left end of the presentation of the function. We saw this in **[Example 2]**, above.

$$f(x) = 6x^5 \underbrace{- 4x^8}_{} + 11x$$

← the leading term is here

Indeed, remember that addition is commutative, so the terms of a polynomial can be relisted in any order (as long as the correct signs are kept attached to the coefficients), without changing the function.

$$f(x) = \underbrace{-4x^8}_{} + 6x^5 + 11x = 11x + 6x^5 \underbrace{-4x^8}_{}$$

The degree can be zero. We saw this in **[Example 4]**, above.

$$f(x) = 5$$

non-zero constant polynomial has degree 0.

The constant term can be the number zero. We saw this in **[Example 2]**, above. (In other words, there might not be a constant term appearing in the presentation of the function.)

$$f(x) = 6x^5 - 4x^8 + 11x$$

But the leading coefficient cannot be zero. That is, the function

$$f(x) = 6x^5 - 4x^8 + 11x$$

could be written

$$f(x) = 0x^{12} - 6x^5 - 4x^8 + 11x$$

But that does not change its named parts as a polynomial. That is,

- the leading term is still $-4x^8$
- the degree is still 8
- the leading coefficient is still -4
- the constant term is still 0

End of Review of Polynomial Functions

Now let's get on with the main topic of this video:

The End Behavior of Polynomial Functions

Remember that the phrase *end behavior* refers to the *left end* or *right end* of a graph. We will be interested in describing the end behavior in words and also describing in more abbreviated notation.

Since describing the behavior of the end of a graph involves describing a *trend*, it is natural to describe the end behavior using the terminology of limits.

For example, the right end of a graph is the portion of the graph where the trend in the x values is that

x gets more and more positive, without bound

Abbreviated in words, x goes to infinity

Abbreviated in symbols, $x \longrightarrow \infty$

We will be interested in the behavior of the graph:

Does the end of the graph go up, or down, or does it level off?

In other words, we will be interested in identifying the trend in the *y* values. More precisely, for the right end of the graph, we will be interested in identifying

the limit, as x goes to infinity, of f(x)

Abbreviated in symbols,

$$\lim_{x \to \infty} f(x)$$

End behavior of Constant Polynomials

The simplest kinds of polynomials, the constant polynomials, have end behavior that is trivial to describe.

For $f(x) = c$, the end behavior could be described as

the y value is constant, at height $y = c$

The corresponding limit description is

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} c = c$$

End behavior of Non- Constant Polynomials

Remember that for non-constant polynomial functions, the ends of the graph always go up or down, depending on the *degree* of the polynomial and the sign of the *leading coefficient*.

<u>End Behavior of non-constant Polynomial Graphs</u>

	positive leading coefficient	negative leading coefficient
even degree	graph goes up on both sides. $\lim\limits_{x\to-\infty} f(x) = \infty$ $\lim\limits_{x\to\infty} f(x) = \infty$	graph goes down on both sides $\lim\limits_{x\to-\infty} f(x) = -\infty$ $\lim\limits_{x\to\infty} f(x) = -\infty$
odd degree	graph goes down left,up on right $\lim\limits_{x\to-\infty} f(x) = -\infty$ $\lim\limits_{x\to\infty} f(x) = \infty$	graph goes up on left, down on right $\lim\limits_{x\to-\infty} f(x) = \infty$ $\lim\limits_{x\to\infty} f(x) = -\infty$

(Remark: There were some typos in this table in the Video for Homework H12. The typos are fixed here, in these printed notes, but they will still appear in the video. Sorry about that.)

Let's return to our earlier examples and describe the end behavior, in words and in limit notation.

[Example 1] For the function $f(x) = 6x^5 - 4x^3 + 11x - 13$

- This is an odd degree polynomial with positive leading coefficient
- the end behavior is the graph goes down on the left and up on the right.
- the end behavior described in limit notation is

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty$$

[Example 2] For the function $f(x) = 6x^5 - 4x^8 + 11x$

- This is an even degree polynomial with negative leading coefficient
- the end behavior is the graph goes down on both ends.
- the end behavior described in limit notation is

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty.$$

[Example 3] For the function $f(x) = -3x + 5$ $x = x^1$

- This is an Odd degree polynomial with negative leading coefficient.
- the end behavior is the left end of the graph goes up, the right end goes down.
- the end behavior described in limit notation is

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty$$

[Example 4] For the function $f(x) = 5$

- This is a degree zero polynomial. Constant function.
- the end behavior is both ends of the graph have constant height $y=5$
- the end behavior described in limit notation is

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 5 = 5$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 5 = 5$$

[Example 5] For the function $f(x) = 0$

- This is a constant polynomial (the <u>zero</u> polynomial)
- the end behavior is both ends of the graph have constant height $y=0$.
- the end behavior described in limit notation is

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 0 = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 0 = 0$$

End of Video