End Behavior of Rational Functions

In this video, we will study the *end behavior* of *rational functions*.

Remember that a rational function is a *ratio* of *polynomials*, where the polynomial in the denominator is not the zero polynomial.

We will study three examples of rational functions in this video

$$f(x) = \frac{7x^2 - 42x + 35}{2x^2 - 16x + 30}$$

$$g(x) = \frac{7x^2 - 42x + 35}{2x^3 - 16x^2 + 30x}$$

$$h(x) = \frac{7x^3 - 42x^2 + 35x}{2x^2 - 16x + 30}$$

[Example 1] Find the *end behavior* of the function

$$f(x) = \frac{7x^2 - 42x + 35}{2x^2 - 16x + 30}$$

Solution:

Remember that the phrase *end behavior* refers to the behavior of the left end and the right end of the graph. That is, as $x \to -\infty$, what is the trend in the y values? And what is the trend in the y values as $x \to \infty$?

We investigate by finding $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$

We will start by finding $\lim_{x \to \infty} f(x)$ in order to determine the *right end behavior*.

Observe that the function $f(x)$ can be factored:

$$f(x) = \frac{7x^2 - 42x + 35}{\underbrace{2x^2 - 16x + 30}_{\text{standard form}}} = \frac{7(x^2 - 6x + 5)}{2(x^2 - 8x + 15)} = \frac{7(x-1)(x-5)}{\underbrace{2(x-3)(x-5)}_{\text{factored form}}}$$

Remember that when we found limits of the form $\lim_{x \to c} f(x)$ where the symbol c was a real number constant, we always use the *factored* form of $f(x)$, because it makes the calculations simpler.

But when finding finding $\lim_{x \to \infty} f(x)$, it is the *standard* form of $f(x)$ that is useful. The reason is that when finding $\lim_{x \to \infty} f(x)$, we are to imagine x getting more and more positive without bound. When x is huge, the *leading terms* in the numerator and denominator, $7x^2$ and $2x^2$ are gigantic. The behavior of $f(x)$ is determined by the ratio of these leading terms. So when finding $\lim_{x \to \infty} f(x)$, we should not even bother to factor the function, and just stick with the *standard form*.

So the limit proceeds as follows:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{7x^2-42x+35}{2x^2-16x+30} = \lim_{x\to\infty} \frac{7x^2}{2x^2} = \lim_{x\to\infty} \frac{7}{2} = \frac{7}{2}$$

use Standard form

Keep only the leading terms

Since $x\to\infty$, we know $x\neq 0$, So we can cancel $\frac{x}{x}$

In this limit, we are supposed to imagine x getting more and more positive, without bound, and consider what happens to the values of $\frac{7}{2}$. But since the value of $\frac{7}{2}$ is always the same, we realize that the limit is the number $\frac{7}{2}$.

Sentence description:

As x gets more and more positive, without bound, the y values get closer and closer to $y=\frac{7}{2}$ (and may even equal $\frac{7}{2}$)

Behavior of graph (This is the *right end behavior*)

There is a horizontal asymptote on the right, with line equation $y=\frac{7}{2}$.

Now we find $\lim\limits_{x\to-\infty} f(x)$ in order to determine the *left end behavior*.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{7x^2-42x+35}{2x^2-16x+30} = \lim_{x\to-\infty} \frac{7x^2}{2x^2} = \lim_{x\to-\infty} \frac{7}{2} = \frac{7}{2}$$

keep only leading terms

Since $x\to-\infty$, we know that $x\neq 0$ so we can cancel $\frac{x^2}{x^2}$

In this limit, we are supposed to imagine x getting more and more negative, without bound, and consider what happens to the values of $\frac{7}{2}$. But since the value of $\frac{7}{2}$ is always the same, we realize that the limit is the number $\frac{7}{2}$.

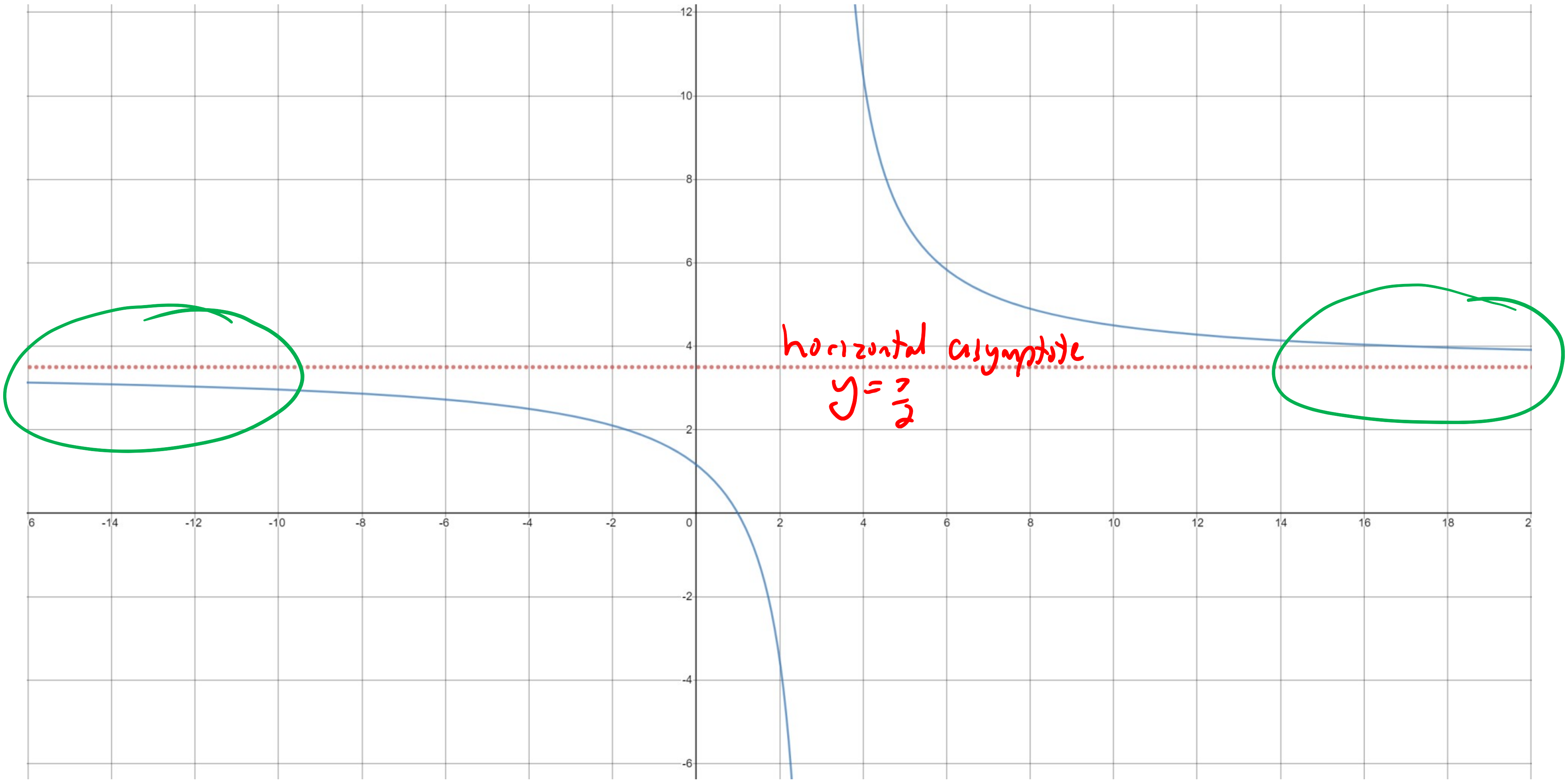
Sentence description:

As x goes to negative infinity, the y values get closer and closer to $y=\frac{7}{2}$

Behavior of graph (This is the *left end behavior*)

There is a horizontal asymptote with line equation $y=\frac{7}{2}$ on the left end of the graph

We can confirm our results with a computer graph.



End of [Example 1]

[Example 2] For the function

$$g(x) = \frac{7x^2 - 42x + 35}{2x^3 - 16x^2 + 30x}$$

find $\lim_{x \to \infty} g(x)$ and $\lim_{x \to -\infty} g(x)$ and explain what the results tell us about the graph of $g(x)$.

Solution:

We start by finding $\lim\limits_{x\to\infty} g(x)$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{7x^2 - 42x + 35}{2x^3 - 16x^2 + 30x} = \lim_{x\to\infty} \frac{7x^2}{2x^3} = \lim_{x\to\infty} \frac{7}{2x} = 0$$

use standard form

keep only the leading terms

Since $x \to \infty$, we know $x \neq 0$ so we can cancel $\frac{x^2}{x^2}$

In this limit, we are supposed to imagine x getting more and more positive, without bound, and consider what happens to the values of $\frac{7}{2x}$. Notice that numerator of the fraction is fixed at 7, while the denominator is getting huge and postive. So the fraction will be getting closer and closer to zero. That is,

as x gets more and more positive, without bound, the value of $\frac{7}{2x}$ will be getting closer and closer to 0.

What does this result tell us about the behavior of the graph?

The right end of the graph has a horizontal asymptote, with line equation $y = 0$.

(right end behavior).

Now we find $\lim_{x \to -\infty} g(x)$

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{7x^2 - 42x + 35}{2x^3 - 16x^2 + 30x} = \lim_{x \to -\infty} \frac{7x^2}{2x^3} = \lim_{x \to -\infty} \frac{7}{2x} = 0$$

use standard form

keep only the leading terms

Since $x \to -\infty$, we know $x \neq 0$ so we can cancel $\frac{x^2}{x^2}$

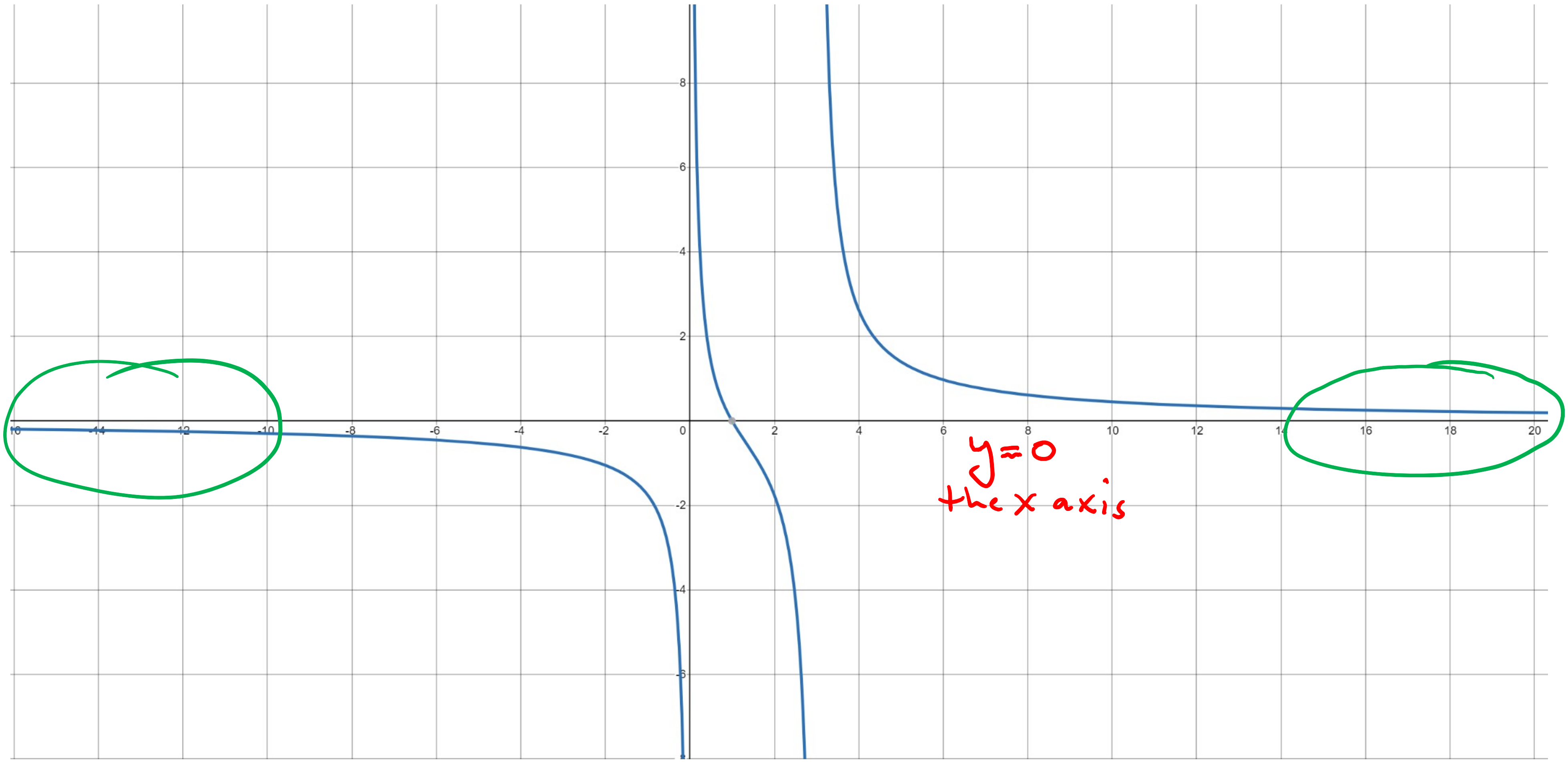
In this limit, we are supposed to imagine x getting more and more negative, without bound, and consider what happens to the values of $\frac{7}{2x}$. Notice that numerator of the fraction is fixed at 7, while the denominator is getting huge and negative. So the fraction will be getting closer and closer to zero. That is,

As $x \to -\infty$, the value of the fraction $\frac{7}{2x}$ will be getting closer and closer to 0.

What does this result tell us about the behavior of graph?

The left end of the graph has a horizontal asymptote with line equation $y = 0$. (left end behavior)

We can confirm our results with a computer graph.



End of [Example 2]

[Example 3] Consider the function

$$h(x) = \frac{7x^3 - 42x^2 + 35x}{2x^2 - 16x + 30}$$

Does the graph have any horizontal asymptotes? If so, give their line equations.

Solution:

Remember that to say that *a graph has a horizontal asymptote* is actually an *abbreviation* for a more detailed description of its end behavior.

Abbreviated description:

The graph has a horizontal asymptote on the right with line equation $y=b$.

More detailed, less abbreviated description:

As x gets more and more positive without bound, the values of $h(x)$ get closer and closer to $y=b$ (and may even equal b)

More abbreviated description, using limit notation:

$$\lim_{x \to \infty} h(x) = b$$ where b is a real number

So in other words, in order to find out if the graph of $h(x)$ has any horizontal asymptotes, we should find $\lim_{x\to\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$

We start by finding $\lim_{x\to\infty} h(x)$ in order to determine the right end behavior

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{7x^3-42x^2+35x}{2x^2-16x+30} = \lim_{x\to\infty} \frac{7x^3}{2x^2} = \lim_{x\to\infty} \frac{7x}{2} = \infty$$

use standard form

keep only the leading terms

Since $x\to\infty$, we know that $x \neq 0$ so we can cancel $\frac{x^2}{x^2}$

In this limit, we are supposed to imagine x getting more and more positive, without bound, and consider what happens to the values of $\frac{7x}{2}$. But if x is huge and positive, then $\frac{7x}{2}$ will also be huge and positive. That is,

As x gets more and more positive, without bound, the value of $\frac{7x}{2}$ also gets more and more positive, without bound.

Limit description: $\lim_{x\to\infty} \frac{7x}{2} = \infty$

What does this result tell us about the behavior of the graph?

The fact that $\lim_{x\to\infty} h(x) = \infty$ tells us that the right end of the graph goes up. There is no horizontal asymptote on the right.

Next, we find $\lim\limits_{x \to -\infty} h(x)$ in order to determine the left end behavior

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{7x^3 - 42x^2 + 35x}{2x^2 - 16x + 30} = \lim_{x\to-\infty} \frac{7x^3}{2x^2} = \lim_{x\to-\infty} \frac{7x}{2} = -\infty$$

use Standard form

keep only leading terms

Since $x \to -\infty$, we know $x \neq 0$, So we can cancel $\frac{x^2}{x^2}$

In this limit, we are supposed to imagine x getting more and more negative, without bound, and consider what happens to the values of $\frac{7x}{2}$. But if x is huge and negative, then $\frac{7x}{2}$ will also be huge and negative. That is,

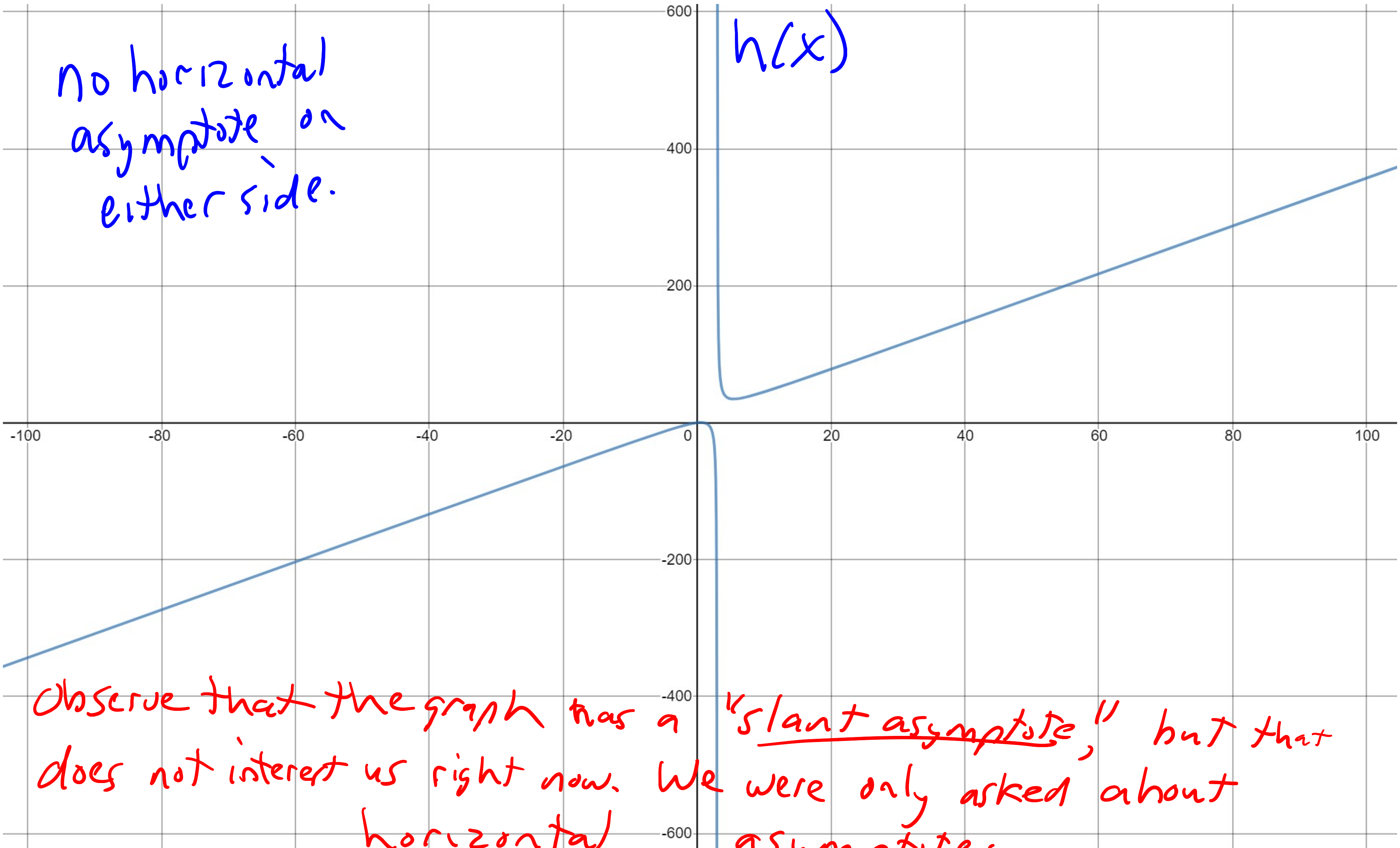
as x gets more and more negative, without bound, the values of $\frac{7x}{2}$ also get more and more negative without bound.

$$\lim_{x\to-\infty} \frac{7x}{2} = -\infty$$

What does this result tell us about the behavior of the graph?

Since $\lim\limits_{x\to-\infty} h(x) = -\infty$, we know that the left end of the graph goes down. So there is no horizontal asymptote on the left.

We can confirm our results with a computer graph.



no horizontal asymptote on either side.

Observe that the graph has a "<u>slant asymptote</u>," but that does not interest us right now. We were only asked about <u>horizontal</u> asymptotes

End of [Example 3]

We can generalize the results of these three examples

Generalization of [Example 1]

If $f(x)$ is a rational function with the degree of the numerator = degree of the denominator, and with leading coefficients a and b in the numerator and denominator, then

limit behavior: $\lim_{x\to\infty} f(x) = \frac{a}{b}$ and $\lim_{x\to-\infty} f(x) = \frac{a}{b}$

end behavior: the graph of f(x) has a horizontal asymptote on both ends, with line equation $y = \frac{a}{b}$

Generalization of [Example 2]

If $g(x)$ is a rational function with the degree of the numerator $<$ degree of the denominator, then

limit behavior: $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = 0$

end behavior: the graph has a horizontal asymptote on both ends, with line equation $y = 0$.

Generalization of [Example ~~1~~ 3]

If $h(x)$ is a rational function with the degree of the numerator $>$ degree of the denominator, then

limit behavior: $\lim_{x \to \infty} h(x) = \infty$ or $-\infty$ and $\lim_{x \to -\infty} h(x) = \infty$ or $-\infty$.

end behavior:

The ends of the graph of $h(x)$ go up or go down.

(No horizontal asymptote).

End of Video