Topic for this Video: Applications of Limits at Infinity: Time going to infinity

Reading:

- **General:** Section 2.2 reading on p. 109 – 114 about limits at infinity.
- **More specifically:** Example 5 on page 112 is the closest particular example

Three concepts from the reading and previous videos that will be important:

Important Concept (1) The definition and graphical significance of limits at infinity

Definition of Limits at Infinity

symbol: $\lim_{x \to \infty} f(x) = b$

spoken: The limit, as x goes to infinity, of $f(x)$ is b.

Usage: x is a variable, f is a function, and b is a real number constant

Meaning: As x gets more and more positive without bound, the values of $f(x)$ get closer and closer to b, and may actually equal b.

Graphical Significance: Graph of f has a horizontal asymptote on the right at $y = b$.
Note that the line equation for the asymptote is $y = b$.

Important Concept (2)

The technique for finding limits at infinity for a rational function

- When finding finding $\lim_{x \to \infty} f(x)$, it is the *standard* form of $f(x)$ that is useful.
- The $\lim_{x \to \infty} f(x)$ is equal to the limit of the ratio of the *leading terms* in the numerator and denominator.

Important Concept (3)

Key results about limits at infinity for a rational function

- If $f(x)$ is a rational function with the degree of the numerator = degree of the denominator, and with leading coefficients a and b in the numerator and denominator, then
  - **limit behavior:** $\lim_{x\to\infty} f(x) = \frac{a}{b}$ and $\lim_{x\to-\infty} f(x) = \frac{a}{b}$
  - **end behavior:** The graph of $f(x)$ has a horizontal asymptote on both sides, with line equation $y = \frac{a}{b}$
- If $f(x)$ is a rational function with the degree of the numerator < degree of the denominator, then
  - **limit behavior:** $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$
  - **end behavior:** The graph of $f(x)$ has a horizontal asymptote on both sides, with line equation $y = 0$
- If $f(x)$ is a rational function with the degree of the numerator > degree of the denominator, then
  - **limit behavior:** $\lim_{x\to-\infty} f(x) = \infty \; or \; -\infty$ and $\lim_{x\to\infty} h(x) = \infty \; or -\infty$
  - **end behavior:** The graph of $f(x)$ goes up or down on the ends. There is no horizontal asymptote.

New terminology in this video:

- The variable will be t, representing ***time***, instead of the more common variable x.
- The questions include a part that asks the student to ***interpret*** a mathematical result. This terminology appears in the Barnett book, and as far as I know, the first appearance of the terminology is in Section 2.2 Example 5. We will discuss it in the examples here.

[Example 1] A drug is administered to a patient by a pill. The drug concentration in the bloodstream is described by the function

$$C(t) = \frac{5t(t+50)}{t^3+100}$$

where t is the time in hours after the pill is taken and $C(t)$ is the drug concentration in the bloodstream (in milligrams/milliliter) at time t.

(a) Find $\lim_{t\to\infty} C(t)$.

(b) Interpret the result. Use appropriate units.

(c) Illustrate the result using a given graph of $C(t)$

(a) Find $\lim_{t\to\infty} C(t)$

Solution:

Convert to standard form: $C(t) = \dfrac{5t(t+50)}{t^3+100} = \dfrac{5t^2+250t}{t^3+100}$

Compute the limit

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{5t^2+250t}{t^3+100} = \lim_{t\to\infty} \frac{5t^2}{t^3} = \lim_{t\to\infty} \frac{5}{t} = 0$$

Keep only the leading terms

Since $t \to \infty$, we know $t \neq 0$, so we can cancel $\frac{t^2}{t^2}$

when you divide 5 by larger and larger numbers, the resulting value gets closer and closer to 0. So the limit is 0.

Remark: This agrees with what we expect

The degree of numerator $= 2 < 3 =$ degree of denominator

So we already knew to expect that $\lim_{t\to\infty} C(t) = 0$

(b) Interpret the result. Use appropriate units.

Solution:

What does the question mean, "Interpret the result"??

The idea is that we have found out a result about the abstract function $C(t)$.

We have found that $\lim_{t \to \infty} C(t) = 0$

But the abstract function $C(t)$ is being used to model a real situation: the concentration of the drug in the bloodstream.

What does the fact that $\lim_{t\to\infty} C(t) = 0$ tell us about the concentration of the drug in the bloodstream?

To answer this, we remind ourselves that the symbol $\lim_{t\to\infty} C(t) = 0$ is an abbreviation for something that can be expressed in a less-abbreviated form, using a sentence. Let's recall that sentence

symbol: $\lim_{t\to\infty} C(t) = 0$

Meaning:

As t gets more and more positive, without bound, the values of C(t) get closer and closer to 0.

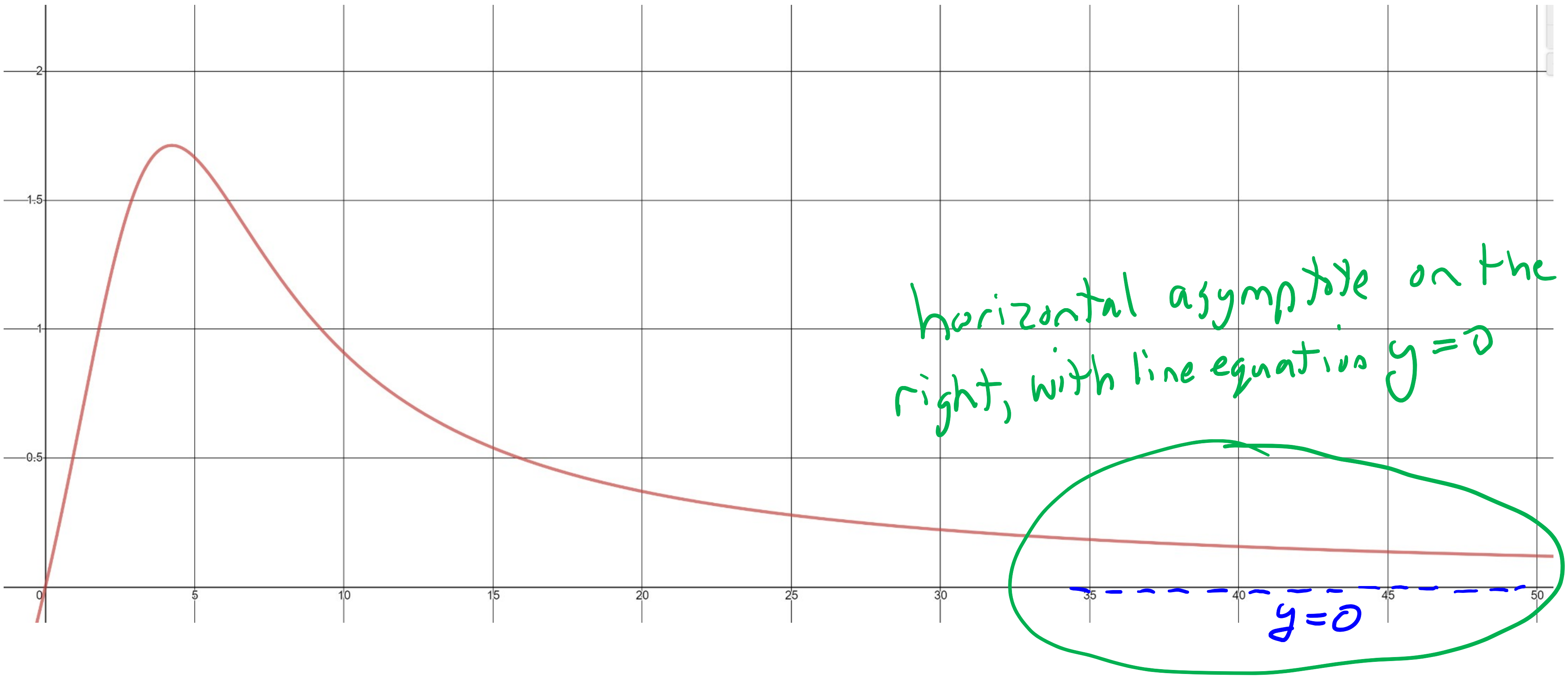
In other words,

As time goes on, the concentration of the drug in the bloodstream decreases and gets closer and closer to 0 milligrams/milliliter.

That is the interpretation.

(c) Illustrate the result using this given graph of $C(t)$

Solution:



End of [Example 1]

[Example 2] A drug is administered to a patient through an IV drip. The drug concentration in the bloodstream is described by the function

$$C(t) = \frac{5t(t^2 + 50)}{t^3 + 100}$$

where t is the time in hours after the pill is taken and $C(t)$ is the drug concentration in the bloodstream (in milligrams/milliliter) at time t.

(a) Find $\lim_{t \to \infty} C(t)$.

(b) Interpret the result. Use appropriate units.

(c) Illustrate the result using a given graph of $C(t)$

(a) Find $\lim_{t\to\infty} C(t)$

Solution:

Get the standard form: $C(t) = \dfrac{5t(t^2+50)}{t^3+100} = \dfrac{5t^3+50t^2}{t^3+100}$

degree of numerator $= 3 =$ degree of denominator

leading coefficient in numerator is 5

leading coefficient in denominator is 1

So $\lim_{t\to\infty} C(t) = \dfrac{5}{1} = 5$

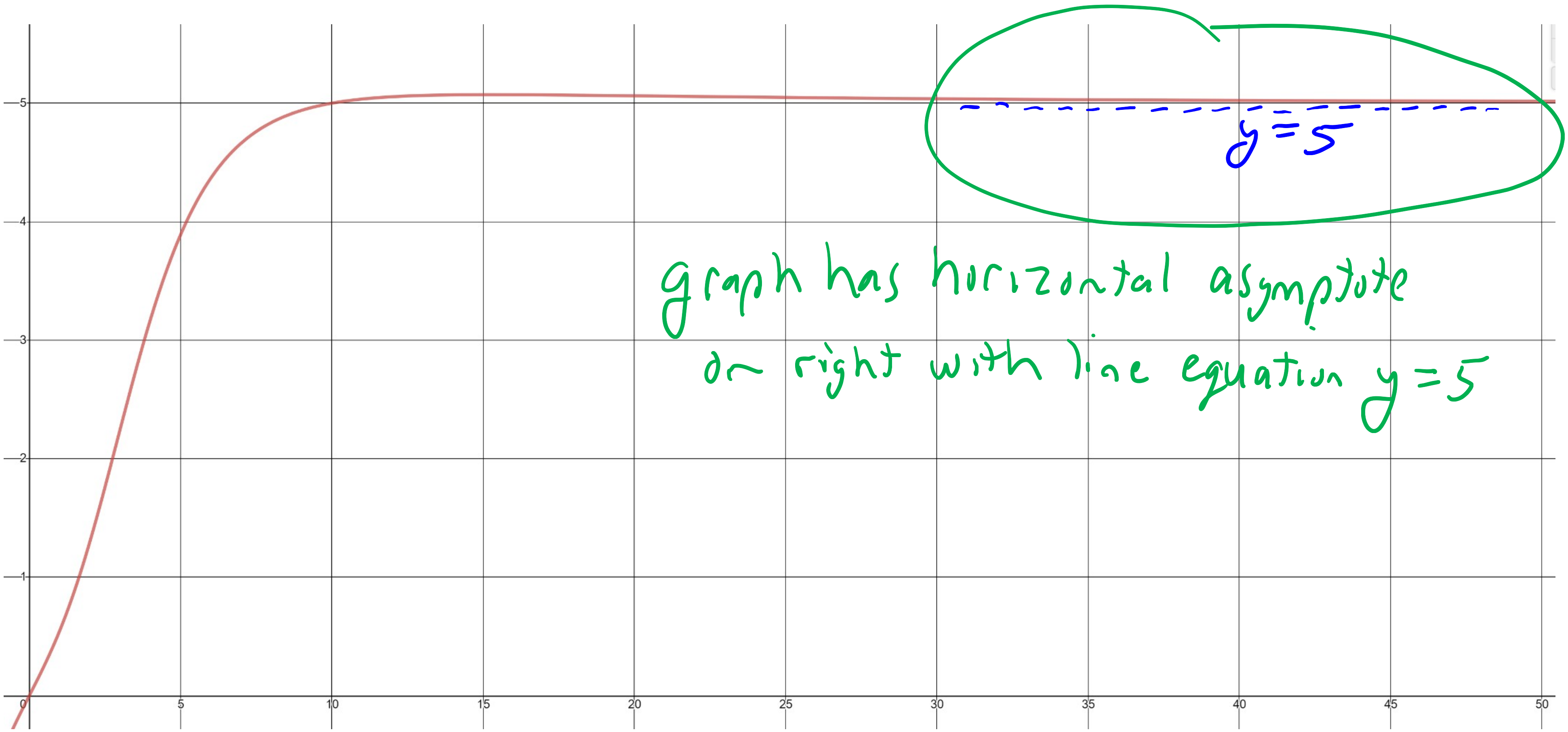
(b) Interpret the result. Use appropriate units.

Solution:

As time goes on, the concentration of drug in the bloodstream levels off at 5 milligrams/milliliter.

(c) Illustrate the result using this given graph of $C(t)$

Solution:



End of [Example 2]

Question: Do these two examples make sense?

Observe that for drug administered by a pill, the mathematical model of **[Example 1]** indicates that the concentration increases to a peak then decreases to zero. This makes sense, because once the pill has dissolved in the stomach and entered the bloodstream, there is no more drug supply. The drug that is used up by the body is not replenished.

For drug administered by IV drip, the model of **[Example 2]** indicates that the concentration increases more slowly and levels off at a non-zero level. This makes sense, because the concentration reaches an equilibrium, where the drug that is used up by the body is continually replenished by the IV. (Of course, this doesn't really go on forever! The mathematical model is really only accurate for the amount of time that the IV is left attached.)

End of Video