Topic for this Video: Find all horizontal & vertical asymptotes

Reading:

- **General:** Section 2.2 reading on p. ~~109 – 114 about limits at infinity~~.
- **More specifically:** Examples 1, 2, 6, 7

Five concepts from the reading and previous videos and prerequisite material that will be important:

Important Concept (1) Correspondence between Asymptotes and Limits

Recall that these words about a horizontal asymptote:

The graph of $f(x)$ has a horizontal asymptote with line equation $y = b$.

Are abbreviated by this symbol: $\lim_{x \to \infty} f(x) = b$, where b is a real number

So to determine whether $f(x)$ has any horizontal asymptotes, we should investigate $\lim_{x \to \infty} f(x)$ to see if it is a real number.

And recall that these words about a vertical asymptote:

The graph of f has a vertical asymptote with line equation $x=c$ and the graph goes up along both sides of the asymptote.

Are abbreviated by this symbol: $\lim\limits_{x \to c} f(x) = \infty$

Of course, there are lots of different ways that a graph can behave at a vertical asymptote. It can go up or down on left side, and up or down on the right side, etc. These kinds of behaviors are abbreviated by symbols such as

$$\lim_{x \to c^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty$$

So to determine whether $f(x)$ has any vertical asymptotes, we should look for real numbers $x = c$ where the $\lim\limits_{\cancel{x \to \infty}} f(x)$ will turn out to be infinite. (Or where the one-sided limits as $x \to c^-$ or $x \to c^+$ will turn out to be infinite.)

typo $x \to c$

Important Concept (2) The form of the function that is most convenient for finding limits.

- When finding finding $\lim_{x\to\infty} f(x)$, it is the *standard* form of $f(x)$ that is useful.
- When finding finding $\lim_{x\to c} f(x)$, it is the *factored* form of $f(x)$ that is useful.

Important Concept (3)

Key results about limits at infinity for a rational function

- If $f(x)$ is a rational function with the degree of the numerator = degree of the denominator, and with leading coefficients a and b in the numerator and denominator, then
  - **limit behavior:** $\lim_{x\to\infty} f(x) = \frac{a}{b}$ and $\lim_{x\to-\infty} f(x) = \frac{a}{b}$
  - **end behavior:** The graph of $f(x)$ has a horizontal asymptote on both sides, with line equation $y = \frac{a}{b}$
- If $f(x)$ is a rational function with the degree of the numerator < degree of the denominator, then
  - **limit behavior:** $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$
  - **end behavior:** The graph of $f(x)$ has a horizontal asymptote on both sides, with line equation $y = 0$
- If $f(x)$ is a rational function with the degree of the numerator > degree of the denominator, then
  - **limit behavior:** $\lim_{x\to-\infty} f(x) = \infty\ or\ -\infty$ and $\lim_{x\to\infty} h(x) = \infty\ or\ -\infty$
  - **end behavior:** The graph of $f(x)$ goes up or down on the ends. There is no horizontal asymptote.

Important Concept (4) Factoring polynomials

Some polynomial functions can be fully factored into linear factors.

For example:

$$f(x) = 2x^2 - 6x - 8 = 2(x^2 - 3x - 4) = 2(x+1)(x-4)$$

or

$$g(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

But some polynomial functions *cannot* be fully factored into linear factors.

For example:

$$h(x) = 3x^2 + 12 = 3(x^2 + 4)$$

The quadratic polynomial $h(x) = 3x^2 + 12$ is called *irreducible*, because it cannot be factored into linear factors.

Important Concept (5)

Correspondence between factors of a rational function and its graph behavior

- If a rational function $f(x)$ has a factor $(x - c)$ appearing in the numerator only, then the graph of $f(x)$ will have an x intercept with coordinates $(c, 0)$.
- If a rational function $f(x)$ has a factor $\frac{(x-c)}{(x-c)}$ appearing in the numerator and denominator with equal powers, then the graph of $f(x)$ will have a hole at $x = c$.
- If a rational function $f(x)$ has a factor $\frac{1}{(x-c)}$ appearing in the denominator denominator only, then the graph of $f(x)$ will have a vertical asymptote with line equation $x = c$.

[Example 1] Find all horizontal and vertical asymptotes for the function:

$$f(x) = \frac{3x^2 - 3x - 36}{2x^2 - 6x - 8}$$

Solution:

To determine whether or not there is a horizontal asymptote, we work with the given *standard* form of $f(x)$

$$f(x) = \frac{3x^2 - 3x - 36}{2x^2 - 6x - 8}$$

We observe that

degree of numerator $= 2 =$ degree of denominator

leading coefficient in numerator is 3

leading coefficient in denominator is 2

Therefore,

Graph has a horizontal asymptote with line equation $y = \frac{3}{2}$.

To determine whether or not there are any vertical asymptotes, we must find the *factored* form of $f(x)$

$$f(x)=\frac{3x^2-3x-36}{2x^2-6x-8}=\frac{3(x^2-x-12)}{2(x^2-3x-4)}=\frac{3(x+3)(x-4)}{2(x+1)(x-4)}$$

We observe that

The factor $(x+1)$ appears in the denominator and not the numerator.

Therefore,

The graph will have a vertical asymptote with line equation $\cancel{y=-1}$.

$x=-1$

mistake in video

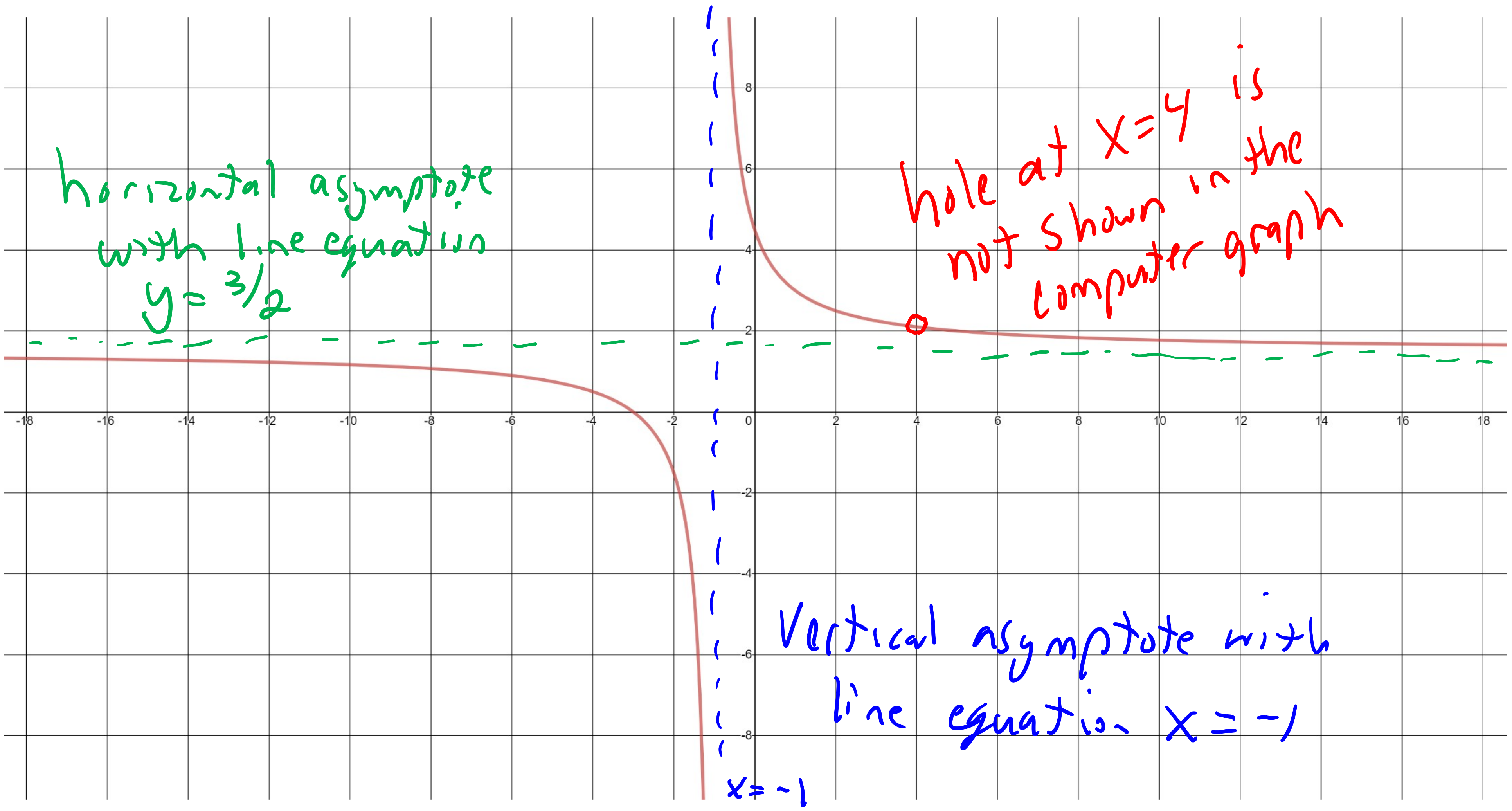
Remark: What do the other factors cause in the graph of $f(x)$?

The factor $(x+3)$ in the numerator and <u>not</u> the denominator will cause an x intercept at $(x,y) = (-3, 0)$

The factor $\dfrac{(x-4)}{(x-4)}$ causes a <u>hole</u> in the graph at $x=4$,

because the factor $(x-4)$ appears in numerator & denominator with equal powers.

We can confirm this with a computer graph



End of [Example 1]

[Example 2] Find all horizontal and vertical asymptotes for the function:

$$f(x) = \frac{5x^3}{3x^2 - 12}$$

Solution:

To determine whether or not there is a horizontal asymptote, we work with the given *standard* form of $f(x)$

$$f(x) = \frac{5x^3}{3x^2 - 12}$$

We observe that

degree of numerator = 3 > 2 = degree of denominator.

Therefore,

There will not be a horizontal asymptote.

To determine whether or not there are any vertical asymptotes, we must find the *factored* form of $f(x)$

$$f(x) = \frac{5x^3}{3x^2-12} = \frac{5x^3}{3(x^2-4)} = \frac{5x^3}{3(x+2)(x-2)}$$

notice difference of two squares

We observe that

The factors $(x+2)$ and $(x-2)$ appear in the denominator and not the numerator.

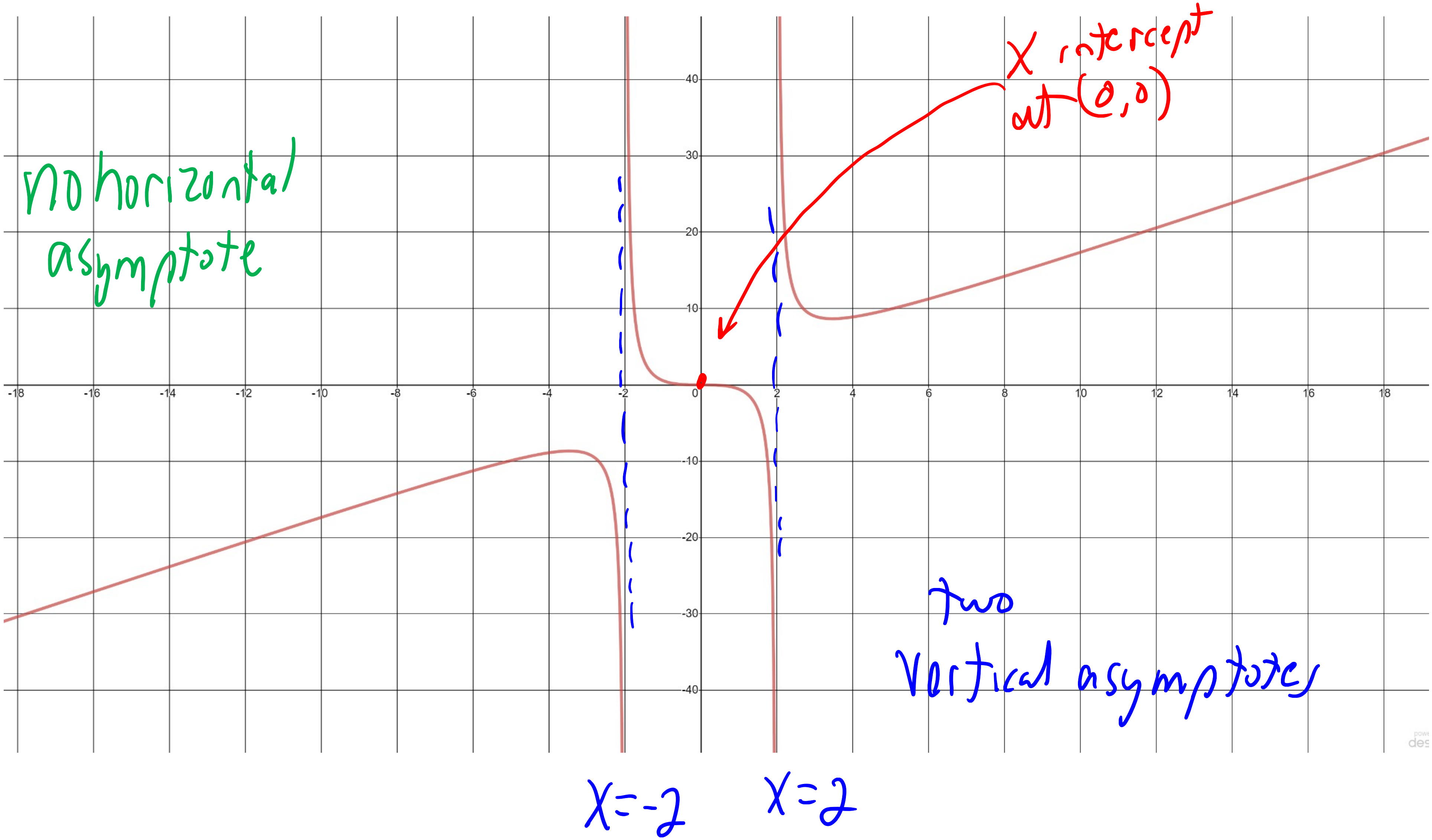
Therefore,

There will be vertical asymptotes with line equations $x=-2$ and $x=2$.

Remark: What do the other factors cause in the graph of $f(x)$?

The factor (x) appears in the numerator alone, raised to the 3rd power. It will cause an x-intercept at $(x,y)=(0,0)$

We can confirm this with a computer graph



End of [Example 2]

[Example 3] Find all horizontal and vertical asymptotes for the function:

$$f(x) = \frac{5x}{3x^2 + 12}$$

Solution:

To determine whether or not there is a horizontal asymptote, we work with the given *standard* form of $f(x)$

$$f(x) = \frac{5x}{3x^2 + 12}$$

We observe that degree of numerator $= 1 < 2 =$ degree of denominator

Therefore,
There will be a horizontal asymptote with line equation $y = 0$

To determine whether or not there are any vertical asymptotes, we must find the *factored* form of $f(x)$

$$f(x) = \frac{5x}{3x^2+12}$$

The denominator cannot be factored further

So the factored form of $f(x)$ is the <u>same</u> as the standard form.

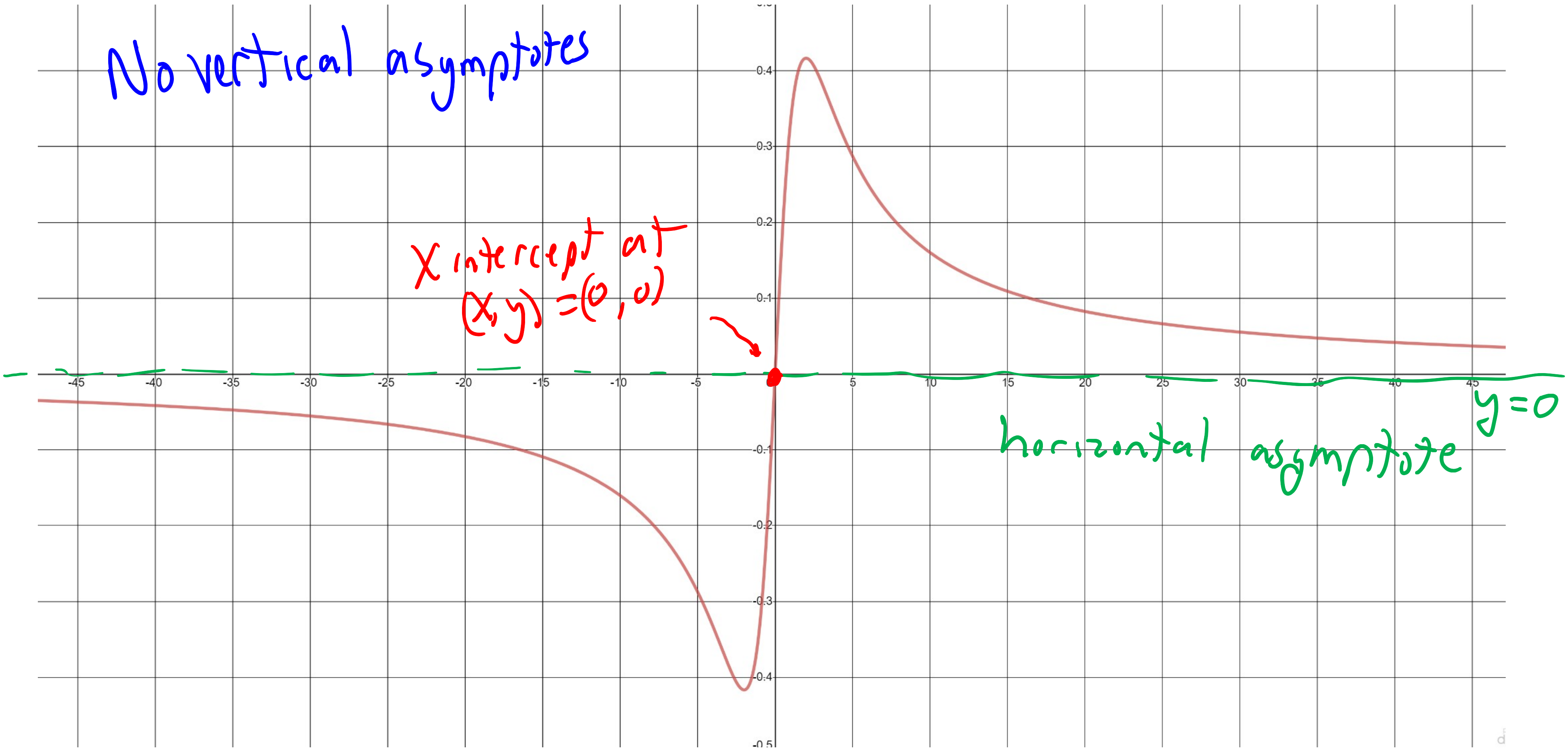
The denominator <u>does</u> <u>not</u> <u>have</u> <u>any</u> linear factors.

So the graph will not have any vertical asymptotes.

Remark: What do the other factors cause in the graph of $f(x)$?

The factor x in the numerator and not the denominator will cause an x intercept at $(x, y) = (0, 0)$

We can confirm this with a computer graph



End of [Example 3]

End of Video