Topic for this video: Given formula for f, where is f continuous?

Reading:

- **General:** Section 2.3 Continuity
- **More specifically:** middle of p. 121 – middle of p. 123, Examples 2, 3

Homework:

H19: Given formula for f, where is f continuous? (2.3#35,37,69)

Background

Recall discussion from Video for H16:

Holes, jumps, and points in the wrong places are easy to spot on a graph. A reasonable question, though, is the following:

> *Is there some way of analyzing the formula for a function to determine if its graph would have any holes & jumps?*

The answer is, *yes*. The concept of *continuity* allows one to analyze the *formula* for a function and determine if (and where) its *graph* would have holes & jumps.

In that video, we saw the introduction of the *definition of continuity*:

Definition of *Continuity at a particular x value*

Words: *The function f is continuous at $x = c$.*

Meaning: the function f passes these three tests:

Test 1: $\lim_{x \to c} f(x)$ exists

Test 1a: $\lim_{x \to c^-} f(x)$ exists

Test 1b: $\lim_{x \to c^+} f(x)$ exists

Test 1c: The numbers in test 1a and 1b agree.

Test 2: $f(c)$ exists

Test 3: The numbers in *Test 1* and *Test 2* agree.

But so far, in Homework H16 and H17 and their videos, we only discussed the continuity of functions by looking at their *graphs*. In this video, we will determine the continuity of functions that are given by *formulas*, not by *graphs*.

For a given function f, we will want to identify the x values where f is *not* continuous, and we will want to then describe the set of all x values where f *is* continuous. There is some terminology that we use that is fairly self-explanatory, but still it is worth presenting in definitions.

Definition of addition terminology involving *continuity*

Words: *The function f is discontinuous at $x = c$.*

Meaning: *The function f is not continuous at $x = c$. That is, it fails the continuity test.*

Words: *The function f is continuous on some set S of x values.*

Meaning: *The function f is continuous at each $x = c$, where the number c is in set S.*

Old tools that we will use: In the examples in this video, and in the associated Homework H19, we will use the terminology and notation of ***intervals***, and ***unions of intervals***, to describe sets of real numbers. That terminology and notation was the subject of Homework H18. (That material is prerequisite material for this course, and so there was no accompanying video.)

New tools that we will use: Continuity Properties from the book Section 2.3

PROPERTIES General Continuity properties

If two functions are continuous on the same interval, then their sum, difference, product, and quotient are continuous on the same interval except for values of x that make a denominator 0.

THEOREM 1 Continuity Properties of Some Specific Functions

(A) A constant function $f(x) = k$, where k is a constant, is continuous for all x.

$f(x) = 7$ is continuous for all x.

(B) For n a positive integer, $f(x) = x^n$ is continuous for all x.

$f(x) = x^5$ is continuous for all x.

(C) A polynomial function is continuous for all x.

$2x^3 - 3x^2 + x - 5$ is continuous for all x.

(D) A rational function is continuous for all x except those values that make a denominator 0.

$\frac{x^2 + 1}{x - 1}$ is continuous for all x except $x = 1$, a value that makes the denominator 0.

(E) For n an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.

$\sqrt[3]{x^2}$ is continuous for all x.

(F) For n an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.

$\sqrt[4]{x}$ is continuous on the interval $[0, \infty)$.

Observations:

It is useful to note that the continuity properties just presented are related to something that we have encountered before. To see the connection, notice that to say that a function f is *continuous at a particular* $x = c$ means that

$$\lim_{x \to c} f(x) = f(c)$$

This single equation embodies the three-part test. That is, in order for the equation to be true,

1) The number $\lim_{x \to c} f(x)$ must exist.
2) The number $f(c)$ must exist.
3) The two numbers must match. That is, $\lim_{x \to c} f(x) = f(c)$.

Recognize that this equation

$$\lim_{x \to c} f(x) = f(c)$$

tells us that when we want to find $\lim_{x \to c} f(x)$ for some function $f(x)$, if we know that $f(x)$ is continuous at $x = c$, then we can find the limit $\lim_{x \to c} f(x)$ by simply computing the y value $f(c)$. This idea, of knowing that we could find certain limits by more simply just computing y values, was the subject of Theorem 2 and Theorem 3 from Section 2.1 of the book, theorems that we used extensively in Homeworks H04, H05, H06, H07. For reference, here is Theorem 3 from Section 2.1

THEOREM 3 Limits of Polynomial and Rational Functions

1. $\lim_{x \to c} f(x) = f(c)$ for f any polynomial function.
2. $\lim_{x \to c} r(x) = r(c)$ for r any rational function with a nonzero denominator at $x = c$.

We will do three examples involving determining the continuity of functions given by formulas.

[Example 1] Determine where the function $f(x) = \dfrac{3x^2 - 3x - 36}{2x^2 - 6x - 8}$ is continuous.

Solution: f(x) is a rational function.

Theorem 1d tells us that it is continuous everywhere <u>except</u> at the x values that cause the denominator to be zero.

So factor f(x)

$$f(x) = \frac{3x^2-3x-36}{2x^2-6x-8} = \frac{3(x^2-x-12)}{2(x^2-3x-4)} = \frac{3(x+3)(x-4)}{2(x+1)(x-4)}$$

f(x) is discontinuous at $x=-1$ and $x=4$, because the denominator is zero there.

f(x) is continuous at all x values <u>except</u> $x=-1$ and $x=4$.

using interval notation

$$\underbrace{(-\infty,-1)}_{\text{intervals}} \underset{\text{union}}{\cup} \underbrace{(-1,4)}_{\text{intervals}} \underset{\text{union}}{\cup} \underbrace{(4,\infty)}_{\text{intervals}}$$

[Example 2] Determine where the function $f(x) = \dfrac{5x}{3x^2+12}$ is continuous.

Theorem 1d tells us that this rational function is continuous everywhere except at the x values that cause the denominator to be zero.

So factor the denominator.

but the denominator is $3x^2+12$, which cannot be factored into linear factors. Since there are no linear factors, there are no x values that will cause the denominator to be zero.

This makes sense, because x^2 is always ≥ 0

so $3x^2 \geq 0$

So $3x^2+12 \geq 12$, so it will never be zero.

Since the denominator is never zero, we conclude that there are no bad x values. So $f(x)$ is continuous at all real numbers.

Presented in interval notation, we would write $\underbrace{(-\infty, \infty)}_{\text{the set of all real numbers.}}$

[Example 3] Let $f(x) = \begin{cases} -2x + 10, & x \leq 3 \\ x^2, & x > 3 \end{cases}$

(A) Graph $f(x)$.

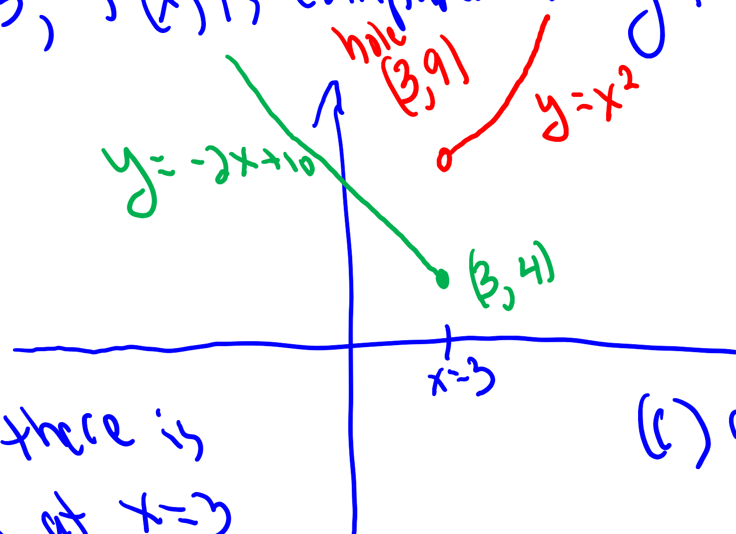
(B) Locate all points of discontinuity.

(C) Find $f(x)$ at all points of discontinuity.

(D) Find $\lim_{x \to c} f(x)$ at every x value c where $f(x)$ is discontinuous.

Solution: We have seen this piecewise-defined function before.

$\begin{cases} \text{when } x \leq 3, \ f(x) \text{ is computed using the formula } f(x) = -2x+10 \\ \text{when } x > 3, \ f(x) \text{ is computed using the formula } f(x) = x^2 \end{cases}$



$f(3) = -2(3) + 10 = 4$

(B) We can see that there is a discontinuity at $x=3$ ('jump' in graph)

(C) at $x=3$, $f(3) = 4$
point on graph at $(x,y) = (3,4)$

(c) Find $\lim_{x \to 3} f(x)$

(i) left limit: $\lim_{x \to 3^-} f(x) = 4$ because graph is heading for the location $(x,y) = (3,4)$ from the left.

(ii) right limit: $\lim_{x \to 3^+} f(x) = 9$ because graph is heading for the location $(3,9)$ from the right.

(iii) Left & right limits don't match, so the $\lim_{x \to 3} f(x)$ does not exist.

This makes sense: the function is not continuous at $x=3$ because it flunks continuity test 1(c)

End of Example
End of Video