Continuing Section 2.3: Continuity

Subject for this video: Sign Behavior of Functions and Solving Inequalities

Reading:

- **General:** Section 2.3 Continuity
- **More specifically:** bottom of p. 123 – top of p. 126, Examples 4, 5

Homework:

H21: Solving Inequalities (2.3#47,49,51,53)

In Calculus, it turns out that an important tool in analyzing a function is to determine where the function is *positive*, *negative*, *zero*, or *discontinuous*. This could generally be described as determining the *sign behavior* of the function.

Tools that we will use that were used in Video for Homework H19:

Continuity Properties from the book Section 2.3

PROPERTIES General Continuity properties

If two functions are continuous on the same interval, then their sum, difference, product, and quotient are continuous on the same interval except for values of x that make a denominator 0.

THEOREM 1 Continuity Properties of Some Specific Functions

(A) A constant function $f(x) = k$, where k is a constant, is continuous for all x.
$f(x) = 7$ is continuous for all x.

(B) For n a positive integer, $f(x) = x^n$ is continuous for all x.
$f(x) = x^5$ is continuous for all x.

(C) A polynomial function is continuous for all x.
$2x^3 - 3x^2 + x - 5$ is continuous for all x.

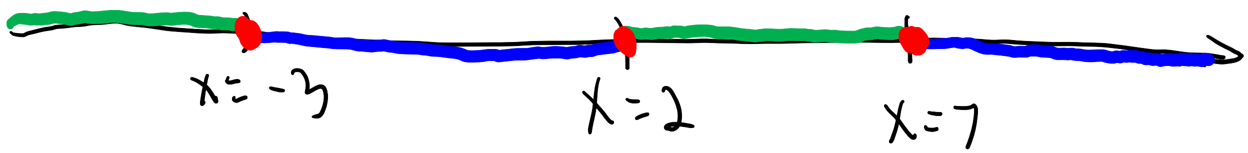
(D) A rational function is continuous for all x except those values that make a denominator 0.
$\frac{x^2 + 1}{x - 1}$ is continuous for all x except $x = 1$, a value that makes the denominator 0.

(E) For n an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
$\sqrt[3]{x^2}$ is continuous for all x.

(F) For n an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.
$\sqrt[4]{x}$ is continuous on the interval $[0, \infty)$.

Observations about sign behavior of functions

Recall observation from previous video: A function that is continuous can only change sign by touching the x axis and crossing it. At those x values, there is an x *intercept*.

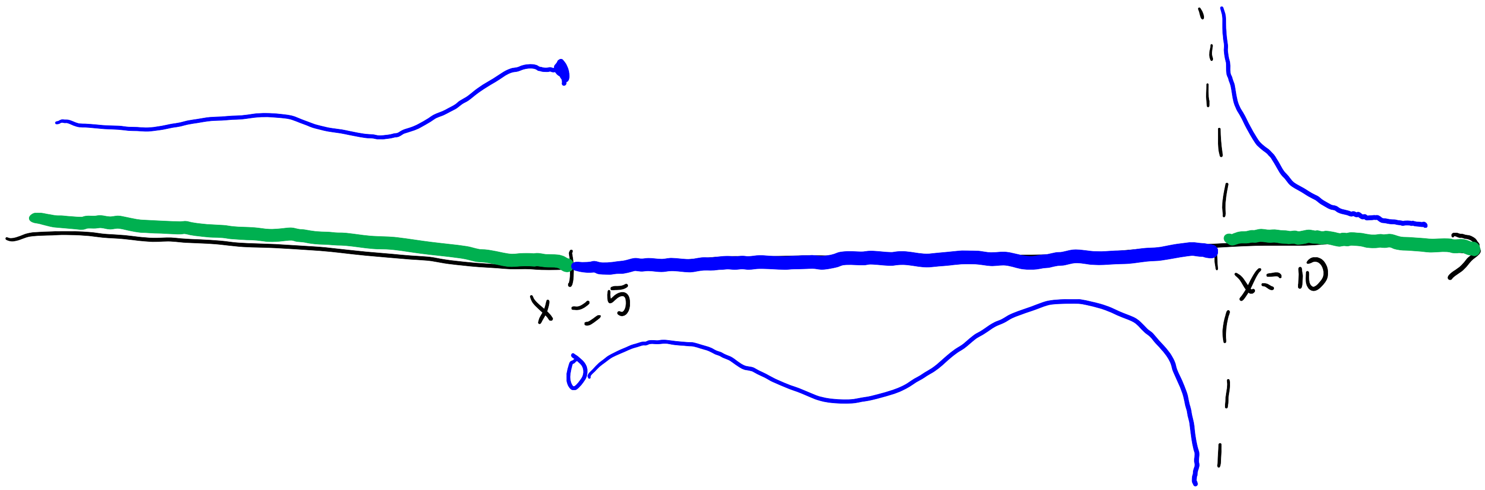


On the intervals *between* those x interecepts, the sign of the function does not change. That is encapsulated in the book's Section 2.3 Theorem 2.

THEOREM 2 Sign Properties on an Interval (a, b)

If f is continuous on (a, b) and $f(x) \neq 0$ for all x in (a, b), then either $f(x) > 0$ for all x in (a, b) or $f(x) < 0$ for all x in (a, b).

But a function that is not always continuous can also change sign by by jumping across the x axis at an x value where the function is discontinuous. At these x values, there is *not* an x intercept.



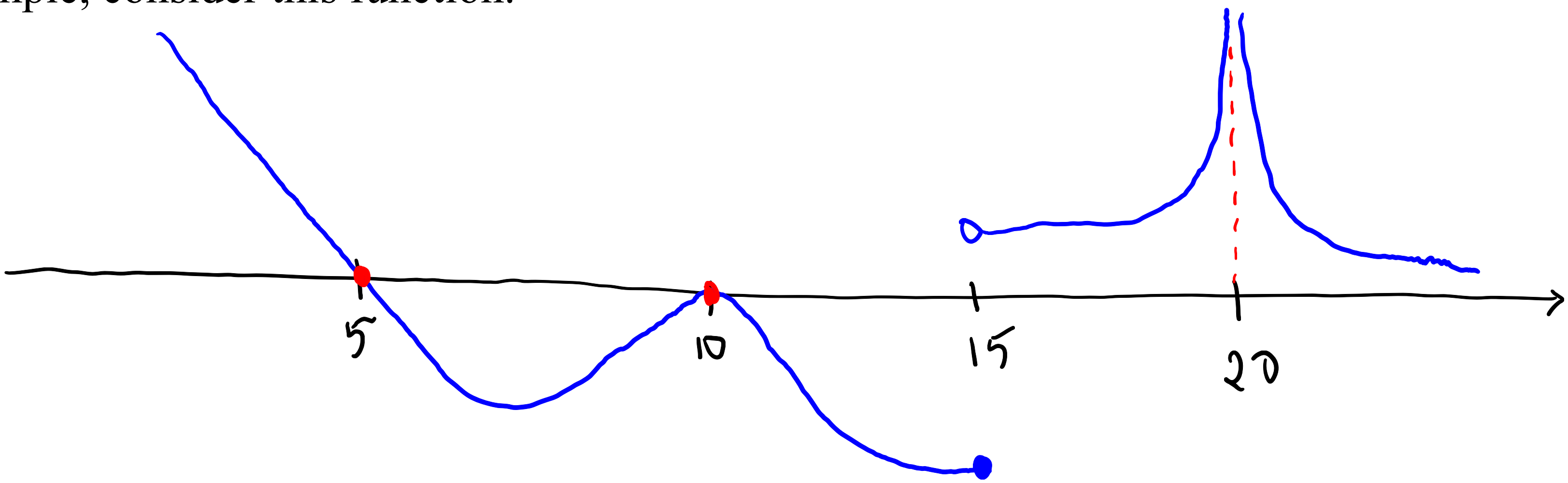
Based on these observations, we see that to determine the sign behavior of a function, it is helpful to first determine the x values where the function has an x intercept or is discontinuous. We will give these x values a name. Here is a definition from the book:

DEFINITION

A real number x is a **partition number** for a function f if f is discontinuous at x or $f(x) = 0$.

On the intervals between the *partition numbers* for f, the sign of the function f does not change.

Keep in mind that a function does not necessarily change sign at *every partition number*. For example, consider this function.



Partition numbers are $x=5, 10, 15, 20$ But f only changes sign at $x=5, 15$

But when looking for x values where a function f *might* change sign, the *only* candidates are the x values that are *partition numbers* for f.

With the terminology of *partition numbers*, we can articulate a procedure for determining the sign behavior of a function using a sign chart. Here is the procedure from the book.

PROCEDURE Constructing Sign Charts

Given a function f,

Step 1 Find all partition numbers of f:

(A) Find all numbers x such that f is discontinuous at x. (Rational functions are discontinuous at values of x that make a denominator 0.)

(B) Find all numbers x such that $f(x) = 0$. (For a rational function, this occurs where the numerator is 0 and the denominator is not 0.)

Step 2 Plot the numbers found in step 1 on a real number line, dividing the number line into intervals. indicate the behavior of $f(x)$ at each partition number.

Step 3 Select a test number in each open interval determined in step 2 and evaluate $f(x)$ at each test number to determine whether $f(x)$ is positive $(+)$ or negative $(-)$ in each interval. Title the diagram: Sign chart for $f(x)$

~~Step 4 Construct a sign chart, using the real number line in step 2. This will show the sign of $f(x)$ on each open interval.~~

We will study four examples involving determining sign behavior for a function by analyzing the formula for the function, and using sign behavior to solve inequalities

[Example 1] (similar to 2.3#49) Let $f(x) = 9x^2 - 90x + 189 = 9(x-3)(x-7)$

standard form — factored form

(a) Determine sign behavior of f.

Solution:

Step 1 find partition numbers

- f is polynomial, so there are no x values that cause f to be discontinuous.
- find x values that cause $f(x) = 0$

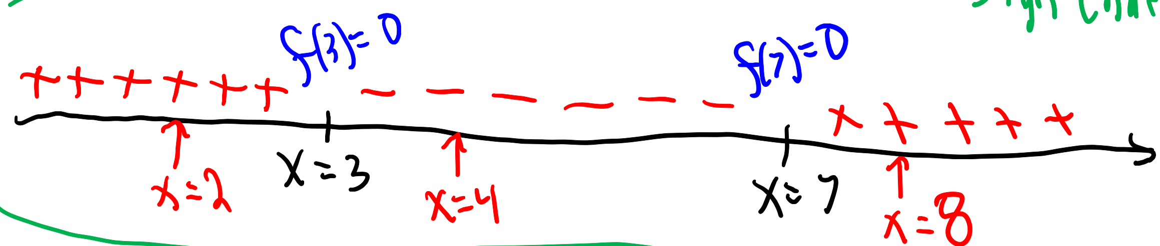
$$0 = 9(x-3)(x-7)$$

Solutions $x=3$, $x=7$ So partition numbers are $x=3, x=7$.

Step 2

Step 3

sign chart for $f(x) = 9(x-3)(x-7)$



$f(2) = 9((2)-3)((2)-7) = 9(-1)(-5) = \text{pos}$

$f(4) = 9((4)-3)((4)-7) = 9(1)(-3) = \text{neg}$

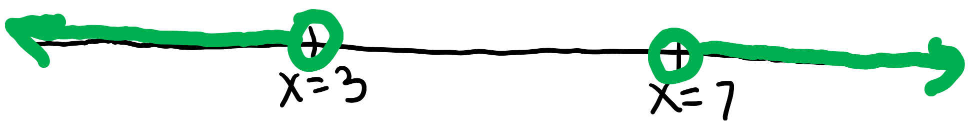
$f(8) = 9((8)-3)((8)-7) = 9(5)(1) = \text{pos}$

(b) Solve the inequality $f(x) > 0$.

Present the solution set 3 ways: Picture Using Number Line, Interval Notation, Set Notation

Book only asks for this

Solution:



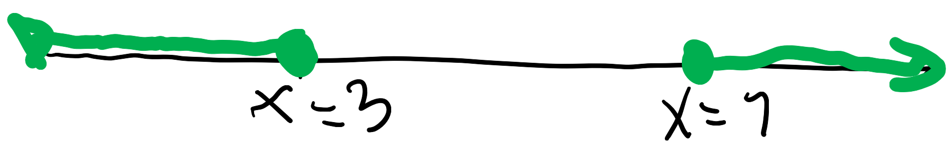
$(-\infty, 3) \cup (7, \infty)$

$\{x$ such that $x < 3$ or $7 < x\}$

(c) Solve the inequality $f(x) \geq 0$.

Present the solution set 3 ways: Picture Using Number Line, Interval Notation, Set Notation

Solution:



$(-\infty, 3] \cup [7, \infty)$

$\{x$ such that $x \leq 3$ or $7 \leq x\}$

[Example 2] (similar to 2.3#49) Solve $9x^2 - 90x < -189$.

Present the solution set 3 ways: Picture Using Number Line, Interval Notation, Set Notation,

Solution:

More helpful form would be an inequality comparing something to 0.

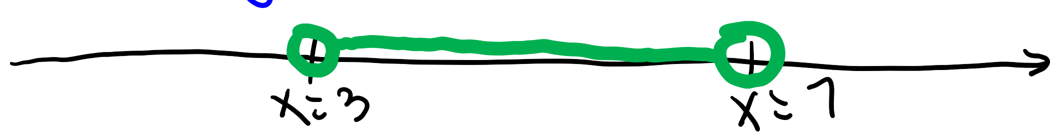
$$9x^2 - 90x < -189$$

add 189 to both sides

$$\underbrace{9x^2 - 90x + 189}_{\text{recognize that this is } f(x) \text{ from previous problem}} < 0$$

We are being asked to solve $f(x) < 0$

Using earlier sign chart



$(3, 7)$

$\{x \text{ such that } 3 < x < 7\}$

[Example 3] (similar to 2.3#51) Solve $x^3 > 49x$.

Present the solution set 3 ways: Picture Using Number Line, Interval Notation, Set Notation,

Solution: Create an inequality where something is compared to 0.

$$x^3 > 49x$$

Subtract 49x from both sides

$$x^3 - 49x > 0$$

Polynomial function $g(x) = x^3 - 49x$

We want to study sign behavior of $g(x) = x^3 - 49x$.

Make a sign chart.

Step 1 Find partition numbers

g is polynomial, so there are no x values that cause g to be discontinuous,

so find x values that cause $g(x) = 0$

$$0 = g(x) = x^3 - 49x = x(x^2 - 7) = x(x+7)(x-7)$$

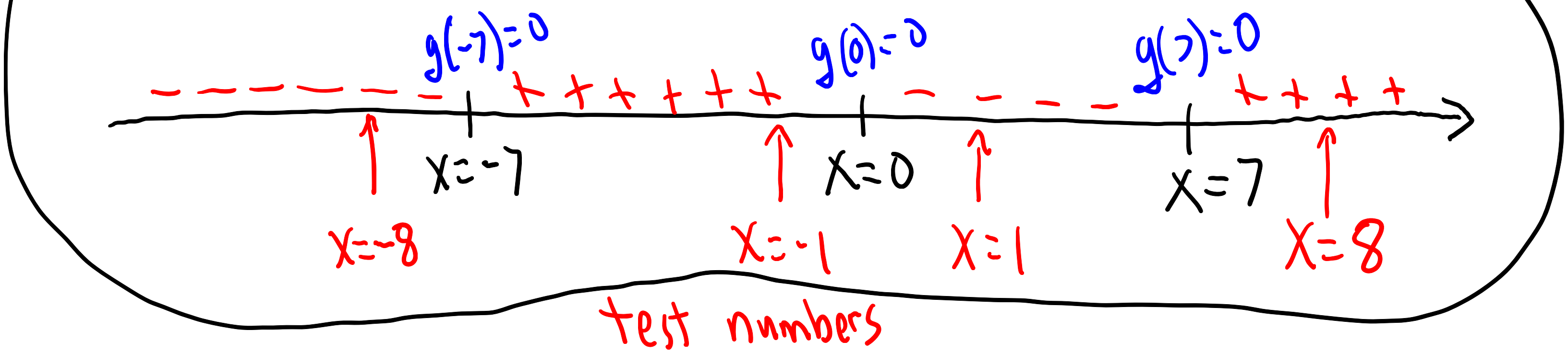
factor

factor some more

$$0 = (x+7)(x)(x-7)$$

Partition numbers $x = -7$, $x = 0$, $x = 7$

sign chart for $g(x) = (x+7)(x)(x-7)$



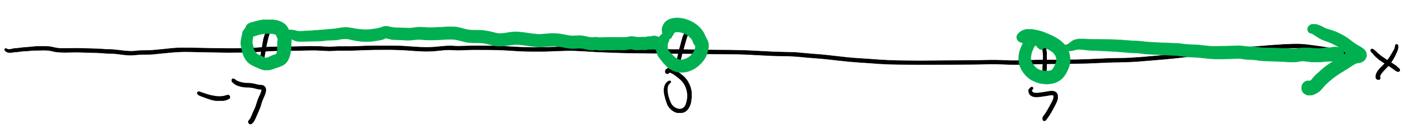
$g(-8) = ((-8)+7)(-8)((-8)-7) = (-1)(-8)(-15) = \text{neg}$

$g(-1) = ((-1)+7)(-1)((-1)-7) = (6)(-1)(-8) = \text{pos}$

$g(1) = ((1)+7)(1)((1)-7) = (8)(1)(-6) = \text{neg}$

$g(8) = ((8)+7)(8)((8)-7) = (15)(8)(1) = \text{pos}$

Our job is to find the x values where $g(x) > 0$



$(-7,0) \cup (7,\infty)$

$\{x \text{ such that } -7 < x < 0 \text{ or } 7 < x\}$

Common incorrect solution to [Example 3]

Common incorrect solution

Solve $x^3 > 49x$

Find partition numbers by solving

$x^3 = 49x$

divide by x ← this step assumes that $x \neq 0$

$x^2 = 49$

$x = 7$ or $x = -7$

incorrect solution

missing $x = 0$

not the correct solution!

[Example 4] (similar to 2.3#53) (a) Solve the inequality $\dfrac{x^2 - 13x}{x - 17} < 0$

Present the solution set 3 ways: Picture Using Number Line, Interval Notation, Set Notation,

Solution:

Identify a function that is involved, and study the sign behavior of that function

Let $f(x) = \dfrac{x^2 - 13x}{x - 17}$. Solve $f(x) < 0$

Make a sign chart for $f(x)$

Step 1 find partition numbers for $f(x) = \dfrac{x^2 - 13x}{x - 17} = \dfrac{x(x-13)}{(x-17)}$

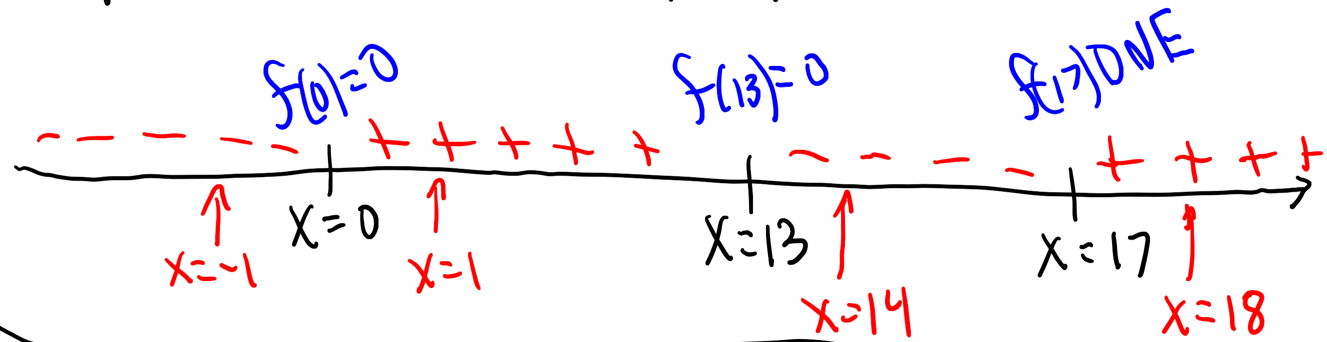
factor

Partition numbers $x = 0$ and $x = 13$ cause $f(x)$ to be zero

$x = 17$ causes $f(x)$ to be discontinuous

Step 2

Sign chart for $f(x) = \dfrac{x(x-13)}{x-17}$



Step 3

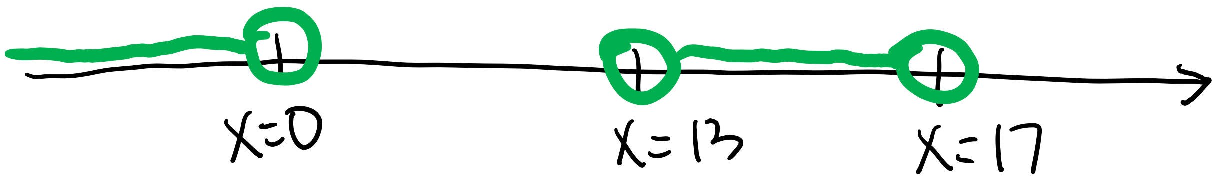
$$f(-1) = \frac{(-1)((-1)-13)}{((-1)-17)} = \frac{\text{neg}\cdot\text{neg}}{\text{neg}} = \text{neg}$$

$$f(1) = \frac{(1)((1)-13)}{((1)-17)} = \frac{\text{pos}\cdot\text{neg}}{\text{neg}} = \text{pos}$$

$$f(14) = \frac{(14)((14)-13)}{((14)-17)} = \frac{\text{pos}\cdot\text{pos}}{\text{neg}} = \text{neg}$$

$$f(18) = \frac{(18)((18)-13)}{((18)-17)} = \frac{\text{pos}\cdot\text{pos}}{\text{pos}} = \text{pos}$$

Solution to inequality $f(x) < 0$ $\quad \dfrac{x^2 - 13x}{x-17} < 0$



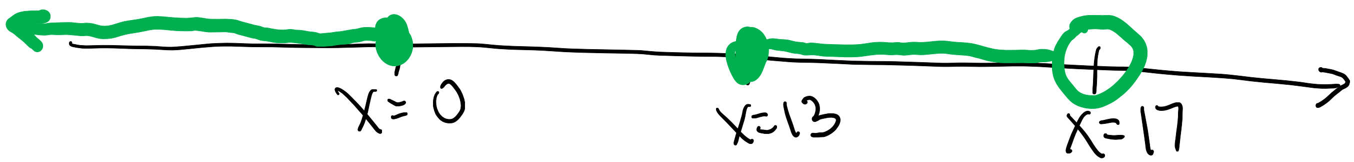
$(-\infty, 0) \cup (13, 17)$

$\{x \text{ such that } x < 0 \text{ or } 13 < x < 17\}$

(b) Solve the inequality $\dfrac{x^2 - 13x}{x - 17} \leq 0$ $f(x) \leq 0$

Present the solution set 3 ways: Picture Using Number Line, Interval Notation, Set Notation,

Solution:



$(-\infty, 0] \cup [13, 17)$

$\{x$ such that $x \leq 0$ or $13 \leq x < 17\}$

[End of Video]