Subject for this video: Secant & Tangent Line Slopes

Reading:

- **General:** Section 2.4 Rates of Change
- **More specifically:** p. 130 – middle of p. 135, Examples 1, 2, 3

Homework:

H25: Secant & Tangent Line Slopes (2.4#9,11,13,45,47)

[Example 1] We will do an extended example involving $f(x) = -x^2 - 4x + 5$

[Example 1] (A) Draw the graph of $f(x) = \underbrace{-x^2 - 4x + 5}_{\text{standard form}}$.

Solution

Standard form tells us

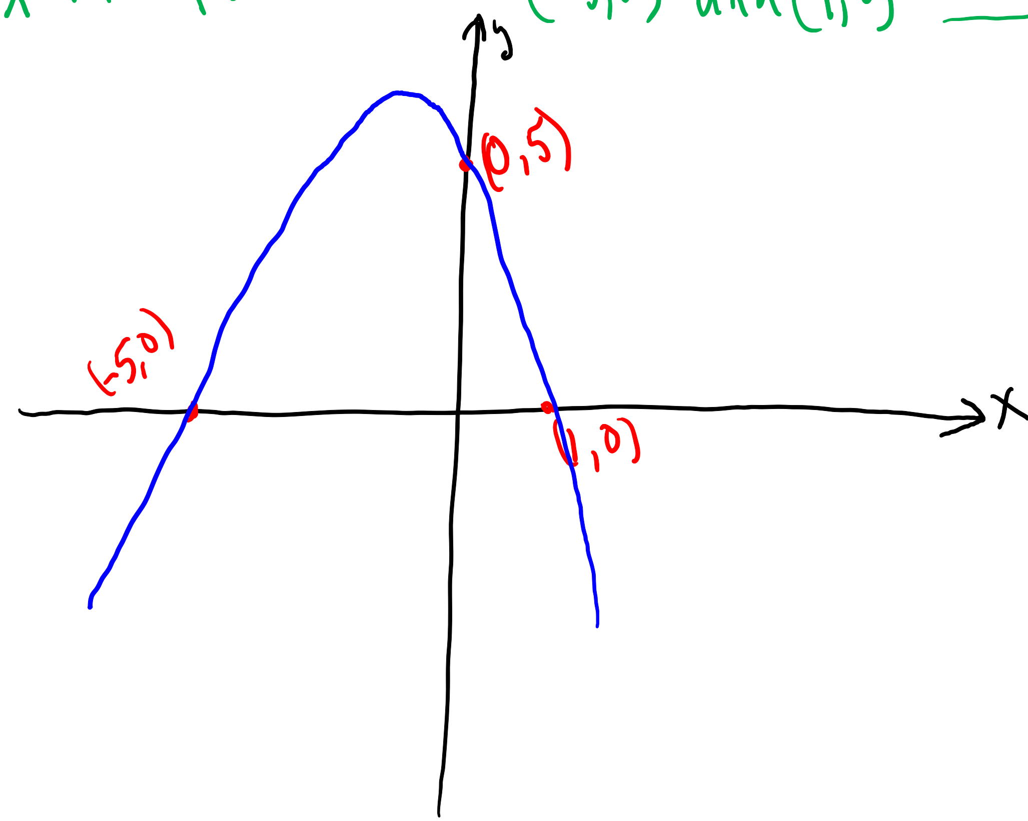
Polynomial with degree = 2 and negative leading coefficient

so graph is a parabola facing down.

y intercept at (0,5)

Get factored form $f(x) = -x^2 - 4x + 5 = -(x^2 + 4x - 5) = -(x+5)(x-1)$

x-intercepts will be at (−5,0) and (1,0)



[Example 1] (B) For the function $f(x) = -x^2 - 4x + 5$, find $f(-3)$ and $f(-1)$ and use that info to label two points on the graph with their (x, y) coordinates.

Solution Start with the empty form of the function

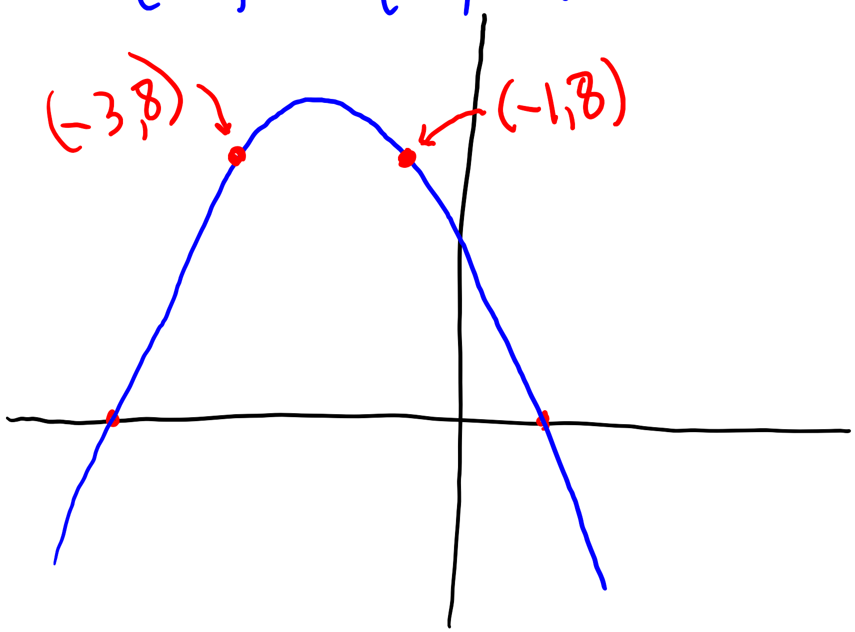
$$f(\ ) = -(\ )^2 - 4(\ ) + 5$$

Plug $x = -3$ into all parentheses

$$f(-3) = -(-3)^2 - 4(-3) + 5 = -(9) + 12 + 5 = 8$$

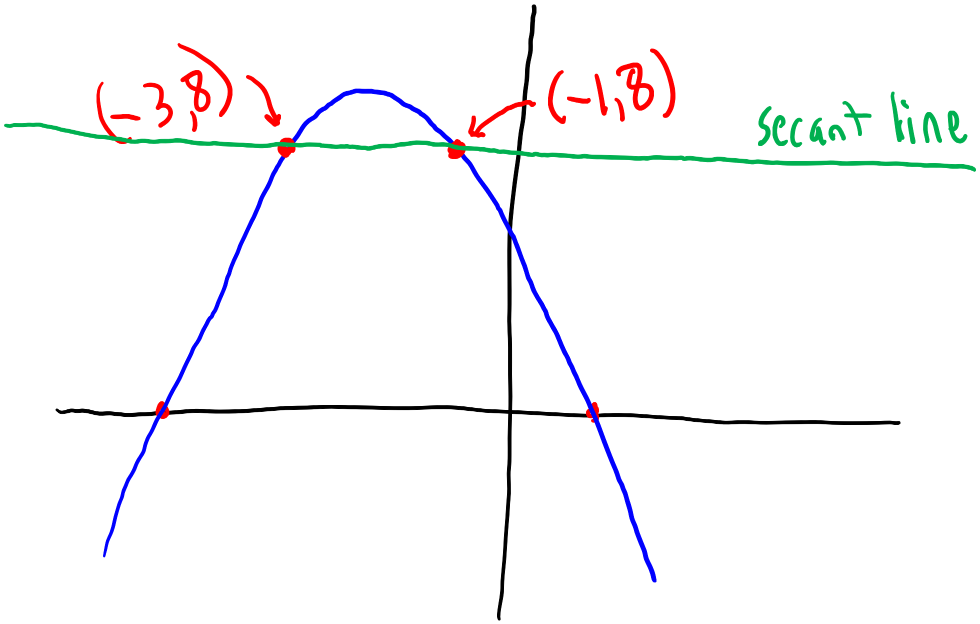
Plug $x = -1$ into all parentheses

$$f(-1) = -(-1)^2 - 4(-1) + 5 = -(1) + 4 + 5 = 8$$



[Example 1] (C) For the function $f(x) = -x^2 - 4x + 5$,

draw the secant line that passes through points $(-3, f(-3))$ and $(-1, f(-1))$ and find its slope.



Slope $m = \frac{\Delta y}{\Delta x} = \frac{8 - 8}{(-1) - (-3)} = \frac{0}{2} = 0$

(horizontal line)

The secant line slope calculation that we just did is related to a more general definition

Definition of *Average Rate of Change*

words: *the **average rate of change** of f from $x = a$ to $x = b$*

usage: f is a function that is continuous on the interval $[a, b]$.

meaning: the number $m = \frac{f(b)-f(a)}{b-a}$

graphical interpretation: The number m is the slope of the ***secant line*** that touches the graph of f at the points $(a, f(a))$ and $(b, f(b))$.

remark: The average rate of change m is a number.

[Example 1] (D) For the function $f(x) = -x^2 - 4x + 5$,

find the *average rate of change of* $f(x)$ *from* $x = -3$ *to* $x = -3 + h$ *where* $h \neq 0$.

1st point

2nd point

Solution

We need to build $m = \dfrac{f(-3+h) - f(-3)}{(\cancel{-3}+h) - (\cancel{-3})} = \dfrac{f(-3+h) - f(-3)}{h}$

We need $f(-3+h)$. Start with the empty version

$f(\quad) = -(\quad)^2 - 4(\quad) + 5$

plug in $-3+h$ into all parentheses

$f(-3+h) = -(-3+h)^2 - 4(-3+h) + 5$

$= -(-3+h)(-3+h) + 12 - 4h + 5$

$= -(9 - 3h - 3h + h^2) + 17 - 4h$

$= -9 + 6h - h^2 + 17 - 4h$

$= 8 + 2h - h^2$

Now use the expressions that we have found to build m.

$$m = \frac{f(-3+h) - f(-3)}{h}$$

$$= \frac{(\cancel{8}+2h-h^2) - (\cancel{8})}{h}$$

$$= \frac{2h-h^2}{h}$$

factor out an h in the numerator

$$= \frac{h(2-h)}{h}$$

since we are told $h \neq 0$, we can cancel $\frac{h}{h}$

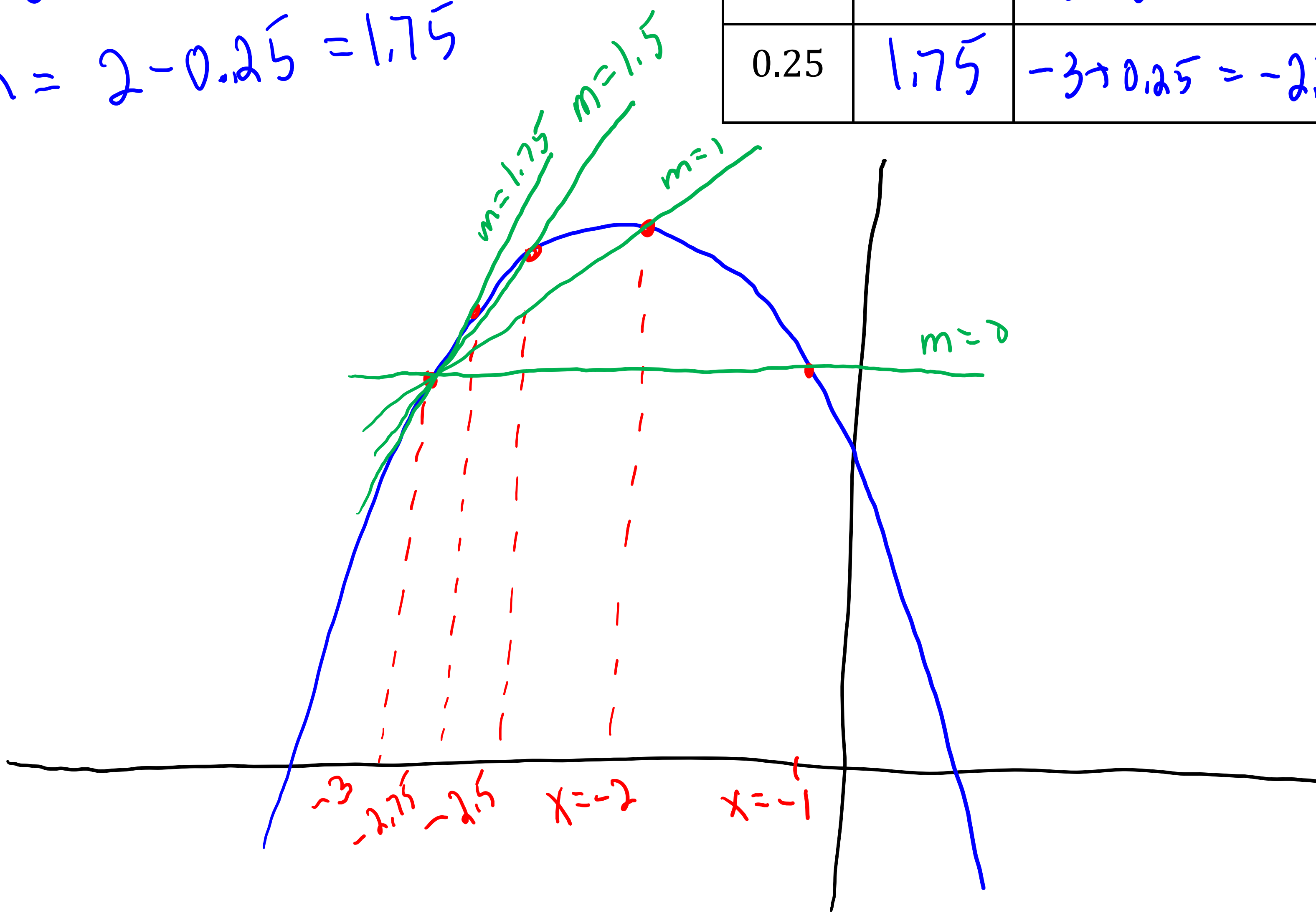
$$m = 2-h$$

Explore the result just obtained by filling in this table and drawing the corresponding secant lines on the graph of f.

h	slope m	x coordinate of 2^{nd} point
2	$m=2-2=0$	$-3+2=-1$
1	$m=2-1=1$	$-3+1=-2$
0.5	1.5	$-3+.5=-2.5$
0.25	1.75	$-3+0.25=-2.75$

$m=2-0.5=1.5$

$m=2-0.25=1.75$



Introduce the *Instantaneous Rate of Change*

Observe that when h gets closer and closer to 0, the 2nd point gets pulled closer and closer to the 1st point and the slopes of the secant lines get closer and closer to the number 2. Remember that the slope numbers are the *average rate of change of* f over some some interval on the x axis. When h gets closer and closer to 0, the x intervals are getting shorter and shorter. If we take the limit of those slopes, as $h \to 0$, the result of that limit is called *the instantaneous rate of change of* f.

Definition of *Instantaneous Rate of Change*

words: the ***instantaneous rate of*** *change of* f *at* a

alternate words: the derivative of f at a

symbol: $f'(a)$

meaning: the number $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

remark: The instantaneous rate of change $f'(a)$ is a number.

[Example 1] (E) For the function $f(x) = -x^2 - 4x + 5$,

find the *instantaneous rate of change of $f(x)$ at $x = -3$*

Solution We need to build

$$m = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}$$

this is the definition of the instantaneous rate of change of $f(x)$ at $x=-3$

use result of part (D)

$$= \lim_{h \to 0} 2 - h$$

limit of polynomial function with variable h.
Theorem 3 tells us that we can substitute $h=0$

$$= 2-(0)$$

$$= 2$$

Introduce the *Tangent Line*

Notice that as $h \to 0$ and the 2nd point gets pulled closer and closer to the 1st point, the secant lines appear to get closer and closer to an imaginary dotted line. We call the imaginary dotted line *the line tangent to the graph of* f *at* $x = -3$. If we had to guess the slope of the imaginary dotted line, a reasonable guess would be the number 2, because that is the number that the secant line slopes are approaching.

The *tangent line* and its slope are defined precisely in the following definition

Definition of the *Tangent Line*

words: *the line tangent to the graph of of* f *at* $x = a$

meaning: the line that has these two properties

- The line contains the point $(x, y) = \big(a, f(a)\big)$, which is called the ***point of tangency***.
- The line has slope $m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, called the ***tangent line slope***.

remark: the *tangent line slope* $m = f'(a)$ is also called the ***slope of the graph at*** $\boldsymbol{x = a}$.

[Example 1] (F) For the function $f(x) = -x^2 - 4x + 5$,

find the *slope of the line tangent to the graph of* $f(x) = -x^2 - 4x + 5$ *at* $x = -3$

Solution: We are being asked to find

$$m = f'(-3) = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} = 2$$

We already did this calculation

The *Equation for the Tangent Line*

Recall the *point slope form* of the equation of a line:

> If a line has these two properties:
>
> - It contains a known point $(x, y) = (a, b)$
> - It has known slope m
>
> Then the ***point slope form*** of the equation of the line is
>
> $$(y - b) = m(x - a)$$

Remember that the line tangent to the graph of f at $x = a$ has the following two properties

- The line contains the point $(x, y) = \big(a, f(a)\big)$, which is called the ***point of tangency***.
- The line has slope $m = f'(a)$, called the ***tangent line slope***.

We can use that information to build the point slope form of the equation for the tangent line

> **The point slope form of the equation for the line tangent to the graph of $\boldsymbol{f(x)}$ at $\boldsymbol{x = a}$**
>
> $$\big(y - f(a)\big) = f'(a) \cdot (x - a)$$

[Example 1] (G) For the function $f(x) = -x^2 - 4x + 5$,

find the equation for the line tangent to the graph of $f(x)$ at ~~$x = 3$~~. $x = -3$

Solution We need to build the equation $(y - \underbrace{f(a)}) = \underbrace{f'(a)}(x - \underbrace{a})$

Get Parts

$a = -3$ the x coordinate of the point of tangency

$f(a) = f(-3) = 8$ the y coordinate of the point of tangency

from earlier part (B) of this [Example]

$f'(a) = f'(-3) = 2$ the slope of the tangent line

from question (F)

Substitute parts into the equation $(y - 8) = 2(x - (-3))$

Convert to slope intercept form by solving for y.

$y - 8 = 2x + 6$

$y = 2x + 6 + 8 = 2x + 14$

Equation for the tangent line.

$y = 2x + 14$

Terminology of *Position* and *Velocity*

Time: When our book uses mathematical functions to describe the motion of objects, x is a variable that represents the elapsed time.

Position: To say an object is *moving in 1 dimension* means that it can go forward or backward in one direction but cannot turn. In such situations, a single coordinate can be used to keep track of the position of the object. A function called the *position function* gives the value of the coordinate at a given time. In our book, the position function is called f. That is, at time x, the coordinate of the object is the number $f(x)$.

average velocity: The words *average velocity from time* $x = a$ *to time* $x = b$ mean the same thing as *average rate of change of position from time* $x = a$ *to time* $x = b$. That is, the number

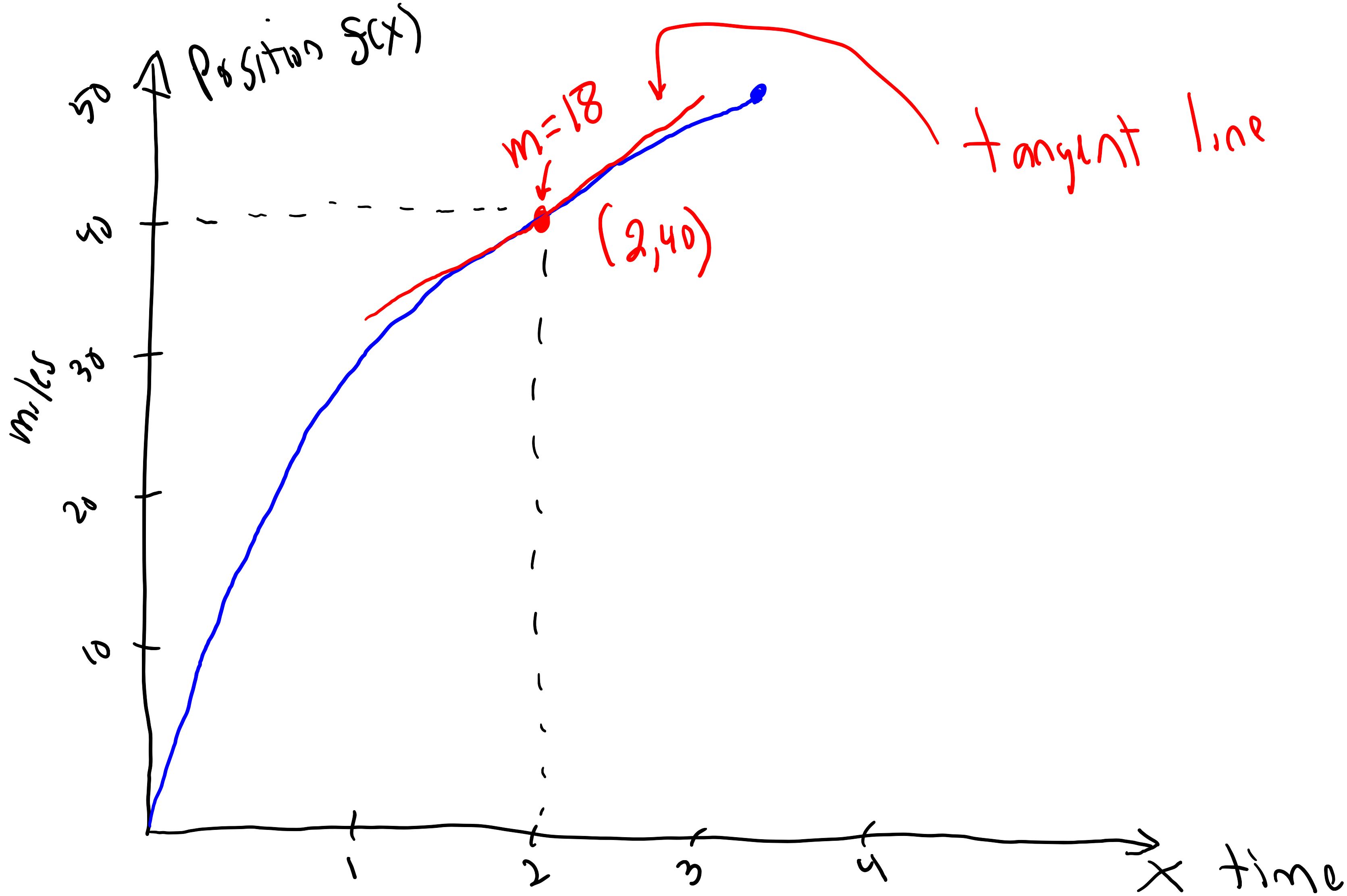
$$m = \frac{f(b) - f(a)}{b - a}$$

instantaneous velocity: The words *instantaneous velocity at time* $x = a$ mean the same thing as *instantaneous rate of change of position at time* $x = a$. That is, the number

$$m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

[Example 2] A cyclist is riding a 50 mile bicycle race. At time 2 hours after the start of the race, the cyclist is at the 40 mile mark and is riding with a velocity of 18 miles per hour.

(A) Let x denote the time (in hours) since the start of the race, and let $f(x)$ represent the distance traveled (in miles) at time x. Draw a graph that illustrates the given information.



(B) Find the cyclist's average velocity during the first two hours of the race, and illustrate your result using the graph.

Solution Average velocity from time $x=0$ to time $x=2$

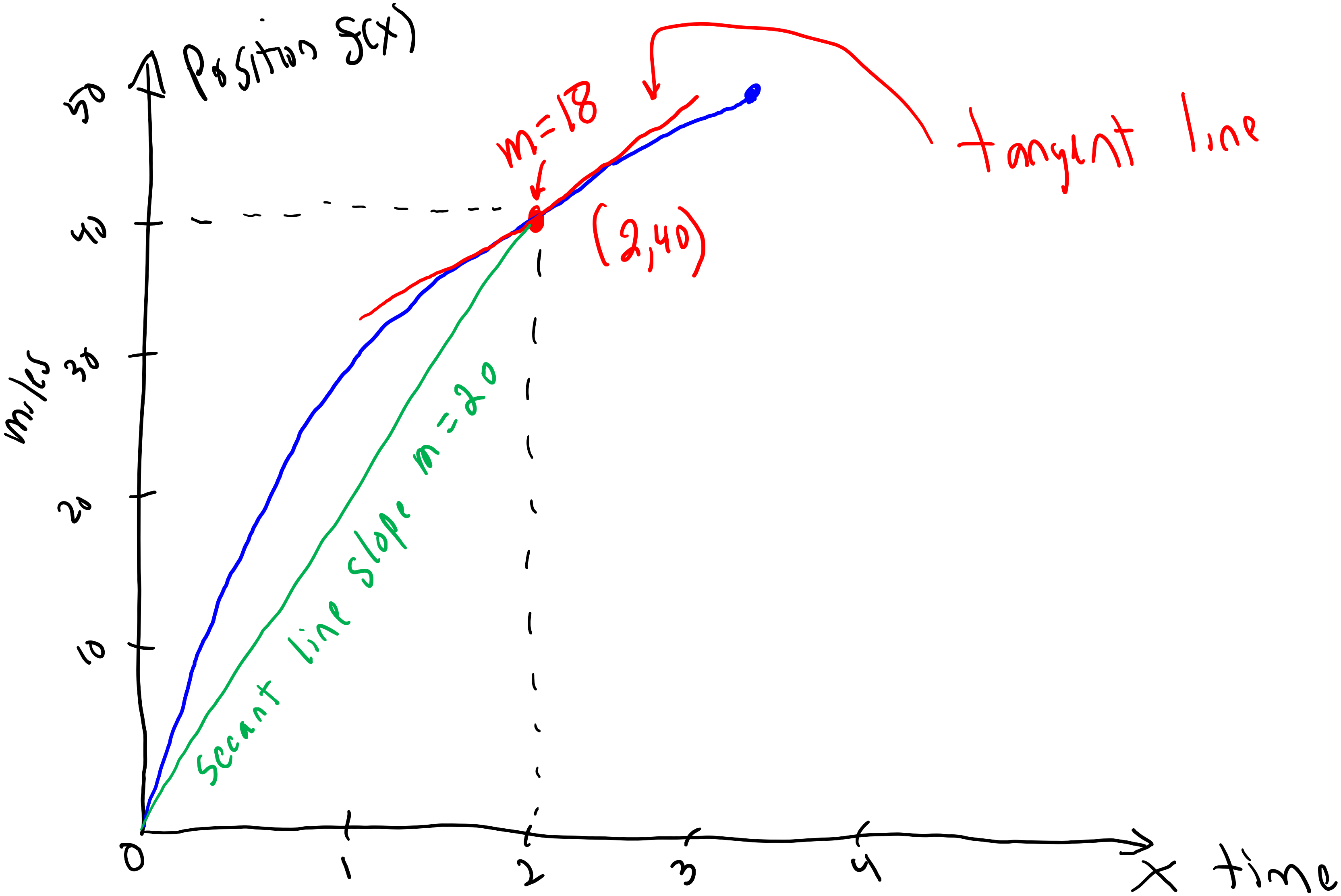
$$= m = \frac{f(2)-f(0)}{2-0}$$

$$= \frac{40-0}{2}$$

$$= 20 \text{ miles per hour}$$

How do we illustrate this result using the graph?

Remember that the Average velocity from time $x=0$ to $x=2$ is the number m that is the slope of the secant line that touches graph at $x=0$ & $x=2$.



Position S(x)
miles
50
40
30
20
10
0
1
2
3
4
x time
m=18
(2,40)
tangent line
secant line slope m=20

(C) Find the equation of the line tangent to the graph of $f(x)$ at $x = 2$.

Solution

We know two things about the tangent line

- it contains the point $(2, 40)$ the point of tangency
- it has slope $m = 18$ (the known velocity)

We can substitute that info into the point slope form of the equation for the tangent line

$$(y - b) = m(x - a)$$

$$(y - 40) = 18(x - 2)$$

Convert to slope intercept form by solving for y

$$y - 40 = 18x - 36$$

$$y = 18x + 4$$

Equation of the tangent line.

Similar Problems on Homework

Your MyLab Math Homework H25 has five problems. Those problems involve calculations very similar to the calculations in **[Example 1]**. and **[Example 2]**. I will discuss those ~~four~~ five homework problems and how they relate to the calculations in the examples.

MyLab Math **Homework H25 problem 1** (book problem 2.4#9)

In part (A), you are asked to compute the value of a difference quotient $\frac{f(b)-f(a)}{b-a}$

A similar calculation was done in **[Example 1](C)** above.

In part (B), you are asked to compute the value of a difference quotient $\frac{f(a+h)-f(a)}{h}$

A similar calculation was done in **[Example 1](D)** above.

In part (C), you are asked to compute the limit of a difference quotient $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

A similar calculation was done in **[Example 1](E)** above.

MyLab Math **Homework H25 problem 2** (book problem 2.4#1)

In part (A), you are asked to compute the ***slope of the secant line*** through point $(a, f(a))$ and $(a+h, f(a+h))$, where $h \neq 0$. Realize that this will involve computing the value of a difference quotient, $m = \frac{f(a+h)-f(a)}{h}$

A similar calculation was done in **[Example 1](D)** above.

In part (B), you are asked to compute the ***slope of a graph*** at a point $(a, f(a))$. That means find the value of $m = f'(a)$. This is obtained by finding the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

A similar calculation was done in **[Example 1](E)** above.

In part (C), you are asked to find the ***equation of the line tangent to a graph*** at a point $(a, f(a))$. Your strategy should be

Get Parts

- Identify the value of a. (This is the x coordinate of *the point of tangency*)
- Find the value of $f(a)$. (This is the y coordinate of *the point of tangency*)
- Identify the value of $f'(a)$. (This is the slope of the tangent line that you got in (B))

Substitute the parts into the *point slope form* of the *equation for the tangent line*

$$\left(y - f(a)\right) = f'(a)(x - a)$$

Convert to *slope intercept form*

Solve the tangent line equation for y to get an equivalent equation of the form

$$y = mx + b$$

A similar calculation was done in **[Example 1](G)** above.

MyLab Math **Homework H25 problem 3** (book problem 2.4#13)

You are asked three questions about a bicycle racer.

A similar problem was done in **[Example 2]** above.

MyLab Math **Homework H25 problem 4** (book problem 2.4#45)

In part (A), you are asked to compute the ***slope of the secant line*** through point $(a, f(a))$ and $(b, f(b))$. That is, find $m = \frac{f(b)-f(a)}{b-a}$ as was done in **[Example 1](C)** above.

In part (B), you are asked to compute the ***slope of the secant line*** through point $(a, f(a))$ and $(a+h, f(a+h))$, where $h \neq 0$. That is, find $m = \frac{f(a+h)-f(a)}{h}$, simlar to **[Example 1](D)**

In part (C), you are asked to compute the ***slope of a graph*** at a point $(a, f(a))$. That means finding $m = f'(a)$. Do this by finding the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, as in **[Example 1](E)** above.

In part (D), you are asked to find the ***equation of the line tangent to a graph*** at a point $(a, f(a))$. A similar calculation was done in **[Example 1](G)** above.

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MyLab Math **Homework H25 problem 5** (book problem 2.4#47)

In part (A), you are asked to find an ***average velocity*** for x changing from a to b. Realize that ***average velocity*** means the same thing ***as average rate of change***. That is, you are being asked to find find $m = \frac{f(b)-f(a)}{b-a}$. A similar calculation was done in **[Example 1](C)** above.

In part (B), you are asked to find an ***average velocity*** for x changing from a to $a + h$. This means find $m = \frac{f(a+h)-f(a)}{h}$, simlar to **[Example 1](D)**

In part (C), you are asked to compute the ***instantaneous velocity*** at $x = a$. Realize that ***instantaneous velocity*** means the same thing ***as instantaneous rate of change***. That is, you are being asked to find $m = f'(a)$. You do this by finding the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, similar to **[Example 1](E)** above.

THE VIDEO ENDS HERE, BUT AN EXTRA EXAMPLE WAS ADDED TO THE NOTES! CONTINUE READING TO FIND THAT EXAMPLE!

[Example 3] (Not presented in the Video) For the function $f(x) = 2x^2$,

(a) Find the *slope of the graph of* $f(x) = 2x^2$ *at* $(5, f(5))$.

(b) The number that is the result of the calculation that you did in part (a) can be interpreted as the *slope* of a tangent line drawn on the graph of $f(x)$. Sketch the graph of $f(x)$. Make your graph large and neat. On your graph, draw the tangent line that corresponds to the slope calculation that you did in part (a). Label the line with its slope by writing

$$m = number$$

where $number$ is the result of your calculation from (a).

(a) Solution: (This solution will be fairly condensed. The concepts involved are discussed at length in **[Example 1]**, presented earlier in this video.)

By the **Definition of the Tangent Line**, presented earlier in this video, we know that the words

$$\textit{slope of the graph of } f(x) = 2x^2 \textit{ at } (5, f(5))$$

mean

$$\textit{slope of the line tangent to the graph of } f(x) = 2x^2 \textit{ at } x = 5$$

Again by the **Definition of the Tangent Line**, we know that this means the number

$$m = f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

Get the parts needed to build the limit expression

$f(x) = 2x^2$ original function

$f(\quad) = 2(\quad)^2$ *empty version*

$f(5) = 2(5)^2 = 2(25) = 50$

$f(5+h) = 2(5+h)^2 = 2\big((5+h)(5+h)\big) = 2(25 + 10h + h^2) = 50 + 20h + h^2$

Put parts into the limit expression and find the limit

$$m = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} \qquad \text{(indeterminate form)}$$

$$= \lim_{h \to 0} \frac{(50 + 20h + h^2) - (50)}{h} \qquad \text{(still indeterminate)}$$

$$\underset{\text{arithmetic}}{=} \lim_{h \to 0} \frac{20h + h^2}{h} \qquad \text{(still indet.)}$$

$$\underset{\text{factor}}{=} \lim_{h \to 0} \frac{h(20+h)}{h} \quad \left(\text{still indet, but since } h \to 0 \text{, we know } h \neq 0 \text{, so we can cancel } \frac{h}{h}\right)$$

$$\underset{\text{cancel}}{=} \lim_{h \to 0} 20 + h \qquad \text{(no longer indeterminate! Now we can use Theorem 3!)}$$

$$\underset{\text{Thm 3}}{=} 20 + (0)$$

$$= 20$$

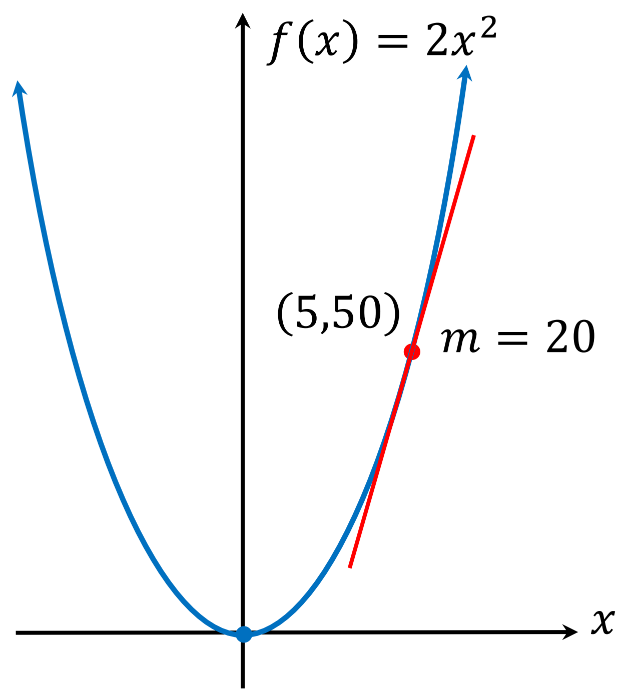
So the slope of of the graph is $m = 20$.

(b) Solution:

The graph of $f(x) = 2x^2$ will be a parabola facing up. We could make a table of (x, y) values and make a precise graph, but it's okay to just hand draw a graph and convey the important stuff:

- The vertex of the parabola will the point $(0,0)$.
- The other important point is $\big(5, f(5)\big)$, which, from part (a), we know is $(5,50)$

We sketch the graph and draw in a tangent line. Label important things.



End of [Example 3]

End of Video.