Subject for this video: Computing $f'(x)$ for a Polynomial Function

Reading:

- **General:** Section 2.4 Rates of Change
- **More specifically:** middle of p. 135 – middle of p. 137, Example 4

Homework:

H26: Computing f' for a Polynomial Function (2.4#19,21,27,29)

In this video, we will learn about the ***derivative*** of a function, and we will study examples involving computing derivatives of polynomial functions.

In the previous video, we discussed the instantaneous rate of change of a function and line tangent to the graph of a function.

Definition of *Instantaneous Rate of Change*

words: the ***instantaneous rate of*** *change of* f *at* a

alternate words: the derivative of f at a

symbol: $f'(a)$

meaning: the number $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

remark: The instantaneous rate of change $f'(a)$ is a number.

Definition of the *Tangent Line*

words: *the line tangent to the graph of of* f *at* $x = a$

meaning: the line that has these two properties

- The line contains the point $(x, y) = (a, f(a))$, which is called the ***point of tangency***.
- The line has slope $m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, called the ***tangent line slope***.

remark: the *tangent line slope* $m = f'(a)$ is also called the ***slope of the graph at*** $\boldsymbol{x = a}$**.**

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Focus on the expression

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In this expression,

- The symbol a represents a number
- The symbol f represents a function
- The symbol $f'(a)$ represents a *number* that is the slope of the line tangent to the graph of the function $f(x)$ at $x = a$. This number is computed by finding a limit

  $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  The symbol $f'(a)$ is spoken *the derivative of* f *at* a.

Now consider replacing the number a with a variable x in all the expressions above. Then the expression

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

would *not* represent a *number*, but rather would represent a *function* of x. That function is called *the derivative of* f.

Definition of the *Derivative*

Symbol: $f'(x)$

Spoken: f *prime of* x

Also spoken: *the derivative of* f *of* x

Meaning: the function $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Used For: When an actual number $x = a$ is substituted into the derivative function $f'(x)$, the resulting number $f'(a)$ can be interpreted as

- the slope of the line tangent to the graph of $f(x)$ at $x = a$
- the slope of the the graph of $f(x)$ at $x = a$
- the instantaneous rate of change of $f(x)$ at $x = a$
- If $f(x)$ is a position function for a moving object, the number $f'(a)$ is the velocity of the object at time $x = a$.

It is important to be careful with terminology when thinking and speaking about derivatives.

One will sometimes hear things like:

The derivative is the slope of the tangent line.

Realize that this is *wrong*. The *slope of the tangent line* is a *number*. The *derivative* is a *function*, *not* a *number*.

It is much better to think of the derivative in the following way:

The derivative is a function that can be used to find the slope of tangent lines.

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In today's video, we will be computing derivatives of polynomial functions using the *Definition of the Derivative* presented two pages ago. That means that, given a function $f(x)$, we will be building the expression for the limit described in the *Definition of the Derivative*,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and then using properties of limits to compute the limit. The result of the limit will be a function that is the derivative function $f'(x)$.

Computing $f'(x)$ using the *Definition of the Derivative* is difficult and messy.

Some of you may have studied calculus before and know about *shortcuts* to finding derivatives. The shortcuts are not nearly so difficult and not nearly so messy.

We will be learning shortcuts in this course, too. They will be called *Derivative Rules*. But that will come later. First, we will spend a few videos discussing (and you will do a few homework sets about) finding derivatives using the *Definition of the Derivative*. That is, the harder, messier method.

In Section 2.4, the book presents a *Four-Step Process* for finding derivatives using the *Definition of the Derivative*. Here it is:

PROCEDURE The four-step process for finding the derivative of a function f:

Step 1 Find $f(x + h)$.

Step 2 Find $f(x + h) - f(x)$.

Step 3 Find $\dfrac{f(x + h) - f(x)}{h}$.

Step 4 Find $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$.

I don't like the book's use of the four-step process, because I feel that it obscures what I think is the most important concept of the first month of the course:

When can one cancel terms, and why?

For that reason, I will use a different procedure

Procedure for finding $f'(x)$ using the *Definition of the Derivative*

Present the goal: to find $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Get parts: Find the expression $f(x+h)$ that will be needed.

Build the limit expression and compute it: Build the expression $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and compute the limit using properties of limits.

Explain clearly:

- Point out limits that are indeterminate forms.
- If cancellation is done, explain clearly why it is justified.

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[Example 1] (Similar to Exercise 2.4#29) Let $f(x) = -3x^2 + 12x + 15$.

(A) Find $f'(x)$ using the ***Definition of the Derivative***.

(B) Find $f'(-2)$.

(C) Find $f'(0)$.

(D) Find $f'(2)$.

(E) Find $f'(4)$.

(F) Illustrate your results from **(B),(C),(D),(E)** using a given graph of $f(x)$.

Remark: The book uses this wording for (A): *Use the four-step process to find* $f'(x)$.

Their question has the same goal, but I prefer to use my own process, not the *four-step process*.

Solution

(A) We need to find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Notice: this is an indeterminate form: If we substitute $h=0$ now, we would get $\frac{f(x+(0)) - f(x)}{(0)} = \frac{f(x) - f(x)}{0} = \frac{0}{0}$

So we can't just compute the limit by substituting $h=0$ right away.

Get parts

We will need $f(x+h)$

$$f(x) = -3x^2 + 12x + 15$$

Get the empty version

$$f(\quad) = -3(\quad)^2 + 12(\quad) + 15$$ empty version

Plug $x+h$ into the parentheses and compute

$$f(x+h) = -3(x+h)^2 + 12(x+h) + 15$$

$$= -3(x+h)(x+h) + 12x + 12h + 15$$

$$= -3(x^2 + xh + xh + h^2) + 12x + 12h + 15$$

$$= -3(x^2 + 2xh + h^2) + 12x + 12h + 15$$

$$= -3x^2 - 6xh - 3h^2 + 12x + 12h + 15$$

Build the limit expression and compute it

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

put in the formula for f(x)

$$= \lim_{h\to 0} \frac{(\cancel{-3x^2} - 6xh - 3h^2 + \cancel{12x} + 12h + \cancel{15}) - (\cancel{-3x^2} + \cancel{12x} + \cancel{15})}{h}$$

cancel terms using arithmetic

$$= \lim_{h\to 0} \frac{-6xh - 3h^2 + 12h}{h}$$

factor out an h in the numerator

$$= \lim_{h\to 0} \frac{h(-6x - 3h + 12)}{h}$$

Still indeterminate form because of the $\frac{h}{h}$

Since $h \to 0$, we know $h \neq 0$, so we can cancel $\frac{h}{h}$

$$= \lim_{h\to 0} -6x - 3h + 12$$

No longer indeterminate!

polynomial in the variable h.

Theorem 3 tells us that we can substitute $h=0$

$$= -6x - 3(0) + 12$$

$$= -6x + 12$$

We found $f'(x) = -6x + 12$

(B) Find $f'(-2)$

Solution This symbol means substitute $x=-2$ into formula for $f'(x)$

$$f'(x) = -6x + 12$$

$$f'(\ \ ) = -6(\ \ ) + 12$$ empty version

$$f'(-2) = -6(-2) + 12 = 24$$

(C) Find $f'(0)$

Solution $f'(0) = -6(0) + 12 = 12$

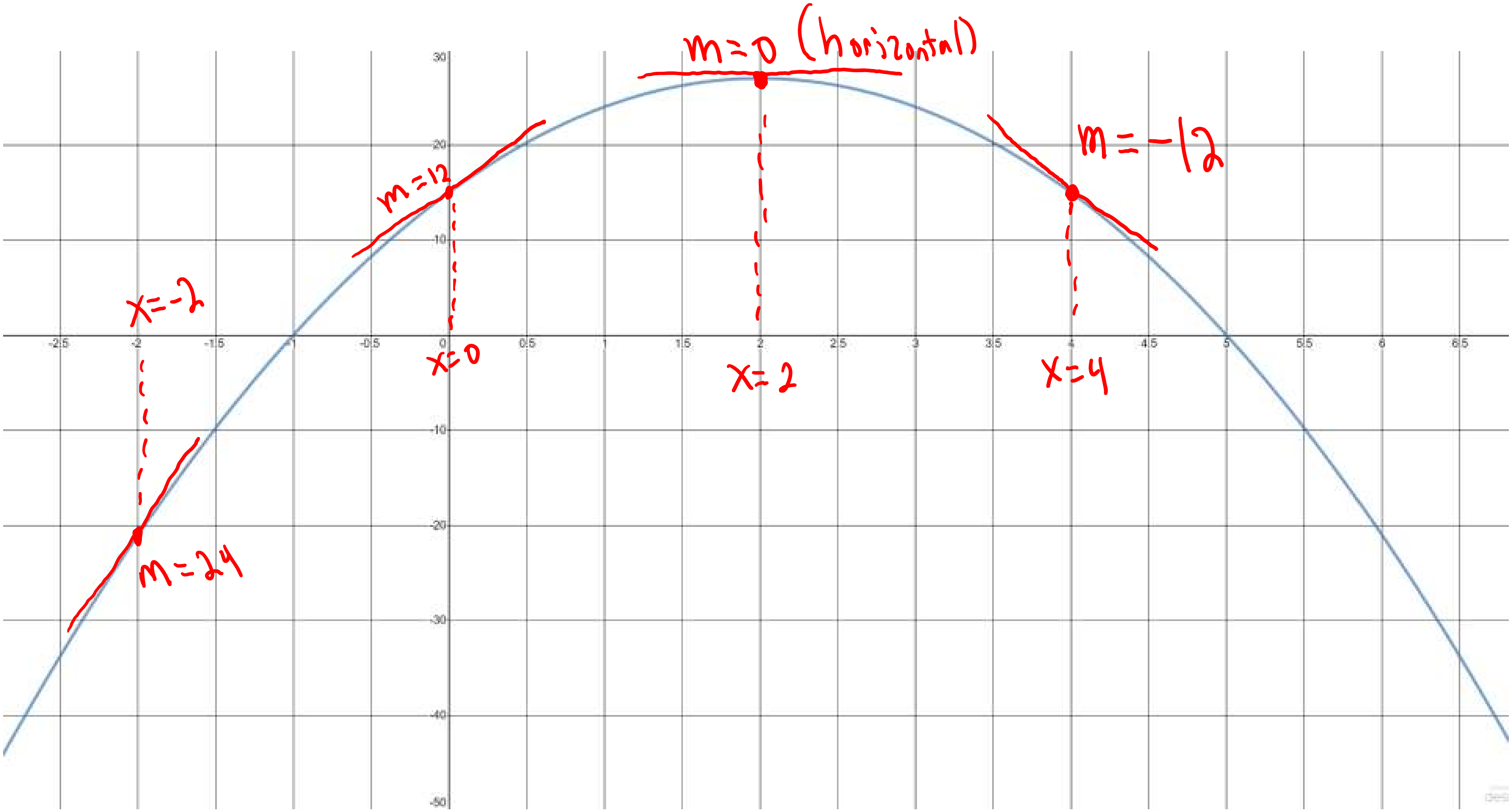
(D) Find $f'(2)$

Solution $f'(2) = -6(2) + 12 = -12 + 12 = 0$

(E) Find $f'(4)$

Solution $f'(4) = -6(4) + 12 = -24 + 12 = -12$

(F) Graph of $f(x) = -3x^2 + 12x + 15$



End of [Example 1]

[Example 2] (Similar to Exercise 2.4#21) Let $f(x) = 5x - 7$.

(A) Find $f'(x)$ using the ***Definition of the Derivative***.

(B) Find $f'(-2)$.

(C) Find $f'(0)$.

(D) Find $f'(2)$.

(E) Find $f'(4)$.

(F) Illustrate your results from **(B),(C),(D),(E)** using a given graph of $f(x)$.

(A) We need to find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Get parts We need $f(x+h)$

$$f(x) = 5x - 7$$

$$f(\quad) = 5(\quad) - 7$$ empty version

$$f(x+h) = 5(x+h) - 7$$

$$= 5x + 5h - 7$$

Put parts into the limit expression and compute the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ indeterminate form

$$= \lim_{h \to 0} \frac{(\cancel{5x} + 5h - \cancel{7}) - (\cancel{5x} - \cancel{7})}{h}$$ indeterminate

cancel using arithmetic

$$= \lim_{h \to 0} \frac{5h}{h}$$ indeterminate

we know $h \to 0$, so $h \neq 0$, so we can cancel $\frac{h}{h}$

$$= \lim_{h \to 0} 5$$

$$= 5$$

We have found that $f'(x) = 5$

(B) find $f'(-2)$

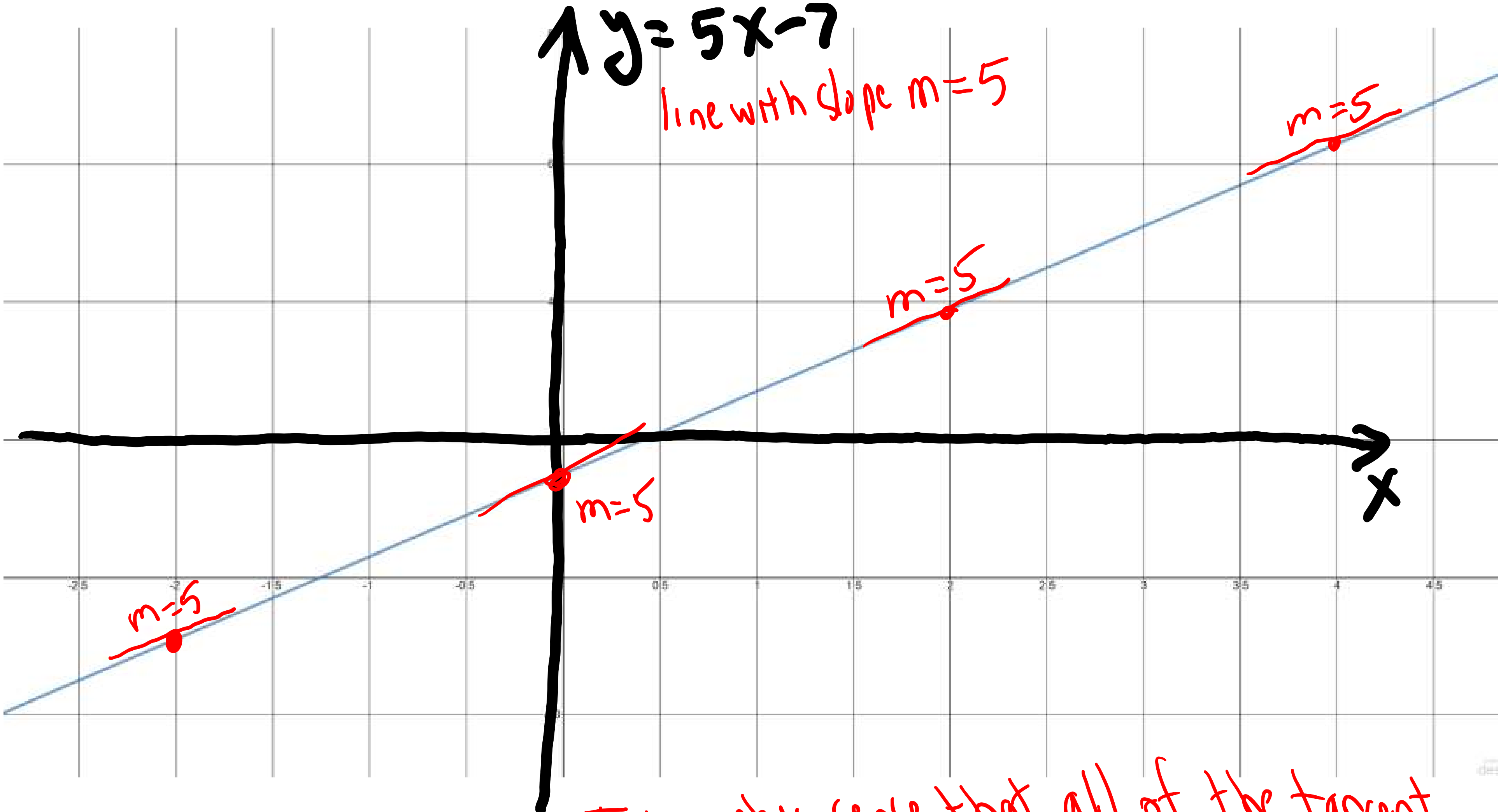
Solution $f'(x) = 5$, so $f'(-2) = 5$

(C) find $f'(0)$ Solution: $f'(0) = 5$

Similarly

(D) $f'(2) = 5$

(E) $f'(4) = 5$



It makes sense that all of the tangent lines have slope m=5 as well.

End of [Example 2]

[Example 3] (Similar to Exercise 2.4#19) Let $f(x) = 7$.

(A) Find $f'(x)$ using the ***Definition of the Derivative***.

(B) Find $f'(-2)$.

(C) Find $f'(0)$.

(D) Find $f'(2)$.

(E) Find $f'(4)$.

(F) Illustrate your results from **(B),(C),(D),(E)** using a given graph of $f(x)$.

(A) We need to find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Get parts   We need $f(x+h)$

$f(x) = 7$

$f(\quad) = 7$ empty version

$f(x+h) = 7$

Build the limit expression and use properties of limits to find the limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{7} - \cancel{7}}{h}$$

Cancel 7 using arithmetic

$$= \lim_{h \to 0} \frac{0}{h}$$

Since $h \to 0$, we know $h \neq 0$, so $\frac{0}{h} = 0$

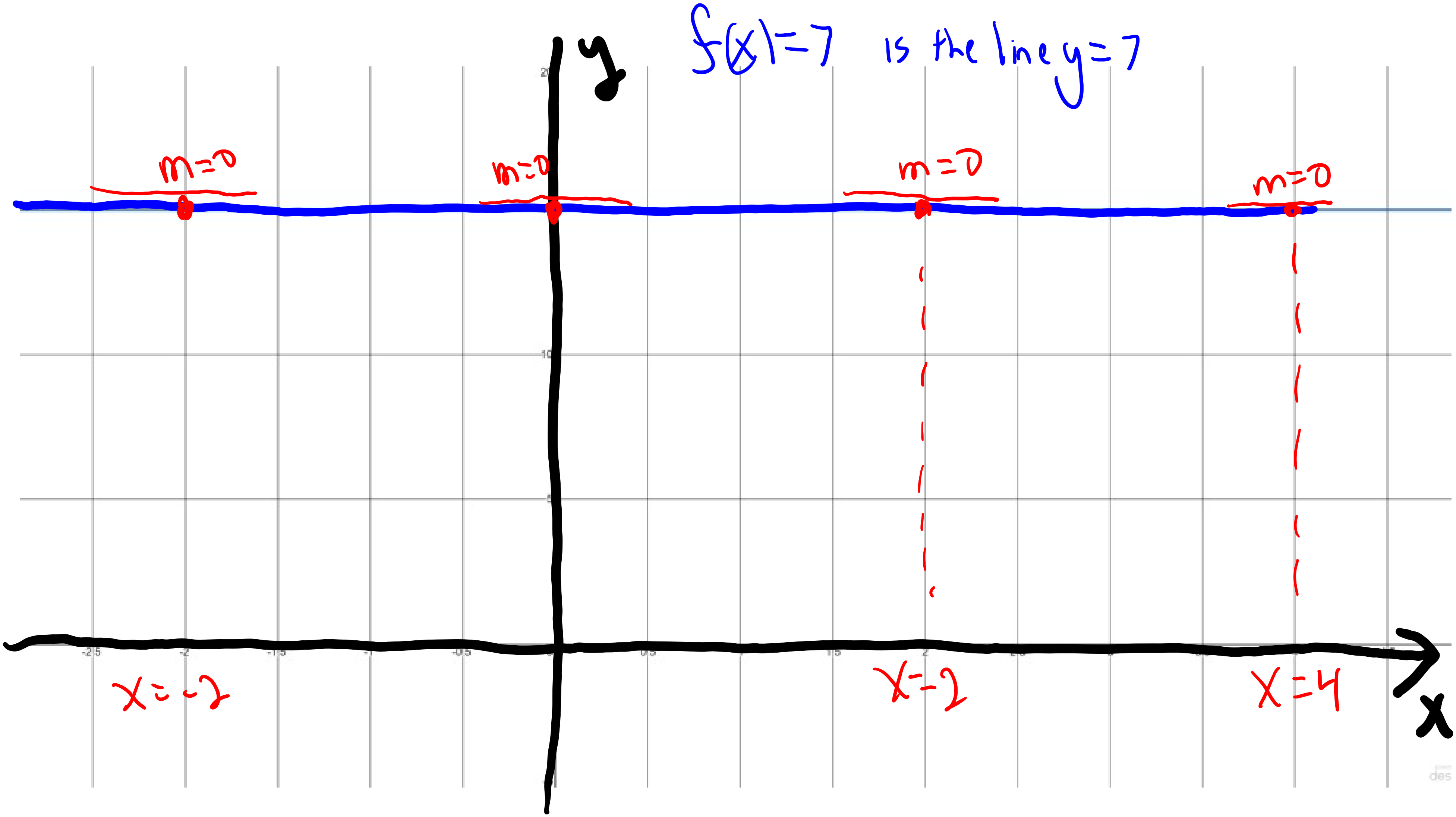
$$= \lim_{h \to 0} 0$$

$$= 0$$

We have found that $f'(x) = 0$

(B), (C), (D), (E)

$$f'(-2) = f'(0) = f'(2) = f'(4) = 0$$



All of the tangent lines are horizontal because the graph of f(x) is horizontal.

End of [Example 3]

End of Video