Subject for this video: Computing $f'(x)$ for $\frac{1}{x}$ and $\sqrt{x}$ type functions

Reading:

- **General:** Section 2.4 The Derivative
- **More specifically:** middle of p. 138 – middle of p. 140, Examples 6,7

Homework:

Computing $f'(x)$ for $\frac{1}{x}$ and $\sqrt{x}$ type functions (2.4#35,37)

Recall the ***Definition of the Derivative***, introduced in Section 2.4 and discussed in the previous video:

Definition of the ***Derivative***

Symbol: $f'(x)$

Spoken: *f prime of x*

Also spoken: *the derivative of f of x*

Meaning: the function $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Used For: When an actual number $x = a$ is substituted into the derivative function $f'(x)$, the resulting number $f'(a)$ can be interpreted as

- the slope of the line tangent to the graph of $f(x)$ at $x = a$
- the slope of the the graph of $f(x)$ at $x = a$
- the instantaneous rate of change of $f(x)$ at $x = a$
- If $f(x)$ is a position function for a moving object, the number $f'(a)$ is the velocity of the object at time $x = a$.

And recall the outline of a process for computing the derivative:

Four Step Process for finding $f'(x)$ using the *Definition of the Derivative*

Present the goal: to find $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Get parts:

Step 1: Find the expression $f(x+h)$

Step 2: Find the expression $f(x+h) - f(x)$

Step 3: Find the expression $\frac{f(x+h)-f(x)}{h}$

Find the limit:

Step 4: Build the expression $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ and compute the limit.

Explain clearly:

- Point out limits that are indeterminate forms.
- If cancellation is done, explain clearly why it is justified.

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[Example 1] (similar to Exercise 2.4#35) Let $f(x) = 7 + \frac{13}{x}$ $\quad \frac{1}{x}$ type function

(A) Find $f'(x)$ using the ***Definition of the Derivative***.

(B) Find $f(1)$ and $f'(1)$.

(C) Find $f(2)$ and $f'(2)$.

(D) Find $f\left(\frac{1}{2}\right)$ and $f'\left(\frac{1}{2}\right)$.

(E) Find $f(-2)$ and $f'(-2)$.

(F) Illustrate your results from **(B),(C),(D),(E)** using a given graph of $f(x)$.

Solution

(A) We need to find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Step 1 find $f(x+h)$

$$f(x) = 7 + \frac{13}{x}$$

$$f(\quad) = 7 + \frac{13}{(\quad)} \qquad \text{empty version}$$

$$f(x+h) = 7 + \frac{13}{(x+h)}$$

Step 2 Find $f(x+h) - f(x)$

$$f(x+h) - f(x) = \left(\cancel{7} + \frac{13}{(x+h)}\right) - \left(\cancel{7} + \frac{13}{x}\right)$$

7 cancels by basic arithmetic

$$= \frac{13}{x+h} - \frac{13}{x}$$

get common denominator

$$= \frac{13x}{(x+h)x} - \frac{13(x+h)}{x(x+h)}$$

$$= \frac{\cancel{13x} - \cancel{13x} - 13h}{x(x+h)}$$

cancel 13x with basic arithmetic

$$= \frac{-13h}{x^2+xh}$$

Step 3 find $\dfrac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{-13h}{x^2+xh}}{h}$$

$$= \frac{1}{h}\cdot\frac{-13h}{x^2+xh}$$

dividing by h is same as multiplying by $\frac{1}{h}$

$$= \frac{-13h}{h(x^2+xh)}$$

Notice: I don't cancel $\underline{h}$ here because we don't know anything about h in this expression.

Later, in step 4, we will know something about h, and I will cancel h there.

Step 4 find $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ indeterminate form

$$= \lim_{h\to 0} \frac{-13h}{h(x^2+xh)}$$ using result from step 3

indeterminate form because of the $\frac{h}{h}$

Since $h \to 0$, we know $h \neq 0$, so we can cancel the $\frac{h}{h}$

$$= \lim_{h\to 0} \frac{-13}{x^2+xh}$$ no longer an indeterminate form!

This is a rational function with variable h.

Notice that $h=0$ is in the domain

So Theorem 3 tells us that we can substitute $h=0$

$$= \frac{-13}{x^2+x(0)}$$

$$= \frac{-13}{x^2}$$

We have found that for $f(x) = 7 + \frac{13}{x}$ the derivative is $f'(x) = \frac{-13}{x^2}$

(B) Find $f(1)$ and $f'(1)$

Solution $f(x) = 7 + \frac{13}{x}$

So $f(1) = 7 + \frac{13}{(1)} = 7 + 13 = \cancel{14}\ 20$

Point on graph at $(x, y) = (1, 20)$

$f'(x) = \frac{-13}{x^2}$

So $f'(1) = \frac{-13}{(1)^2} = -13$

The line tangent to the graph at $x = 1$ has slope $m = -13$

(C) Find $f(2)$ and $f'(2)$

Solution: $f(2) = 7 + \frac{13}{2} = \frac{27}{2}$

Point on graph at $\left(2, \frac{27}{2}\right)$

$f'(2) = \frac{-13}{(2)^2} = \frac{-13}{4}$

Slope of tangent line is $-13/4$

(D) find $f(1/2)$ and $f'(1/2)$

Solution $f(1/2) = 7 + \frac{13}{1/2} = 7 + 13 \cdot 2 = 33$

point at $(1/2, 33)$

$f'(1/2) = \frac{-13}{(1/2)^2} = -13 \cdot 4 = -52$

tangent line Slope is $m = -52$

(E) find $f(-2)$ and $f'(-2)$

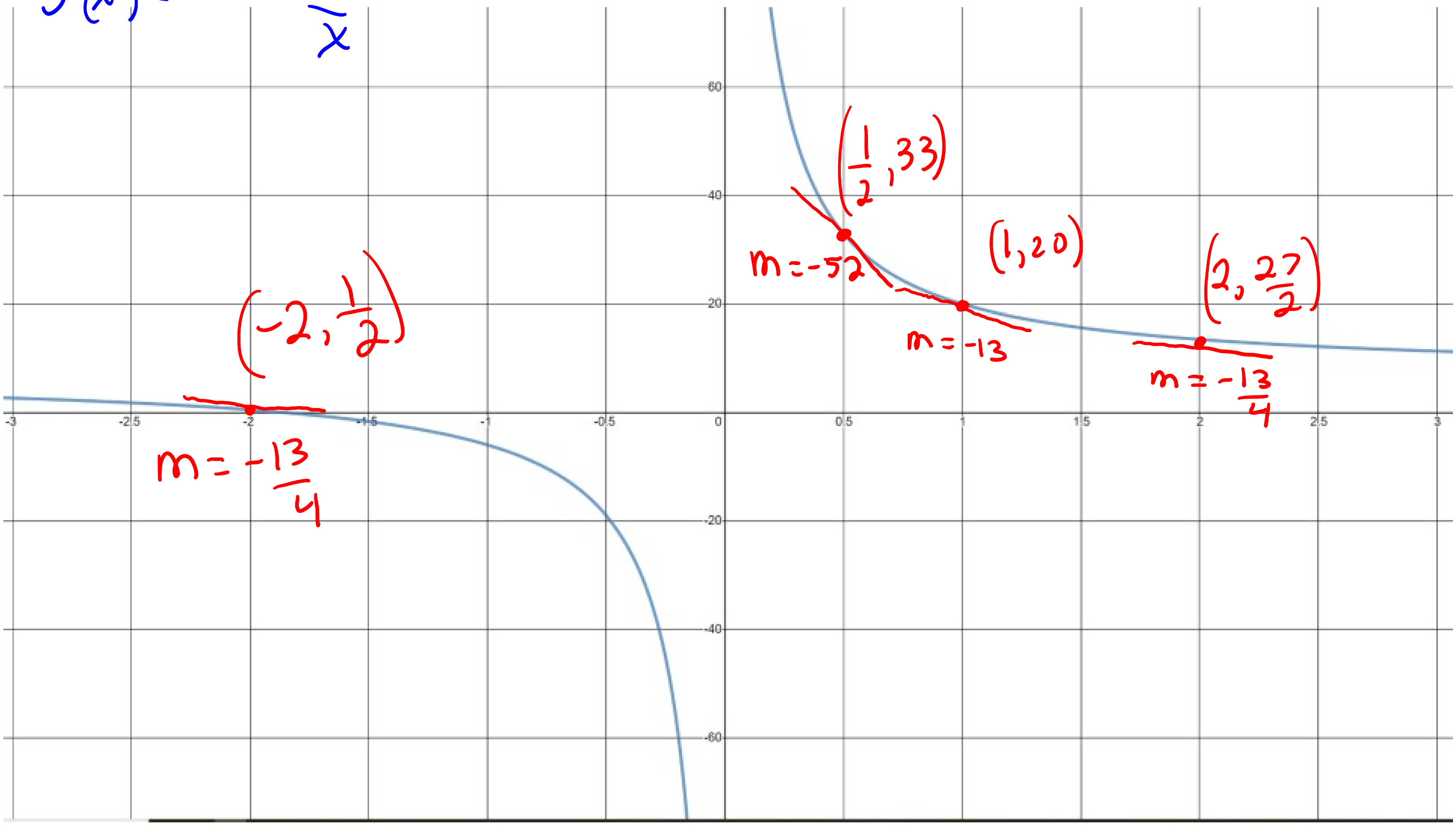
Solution $f(-2) = 7 + \frac{13}{(-2)} = 7 - \frac{13}{2} = \frac{1}{2}$

point at $(-2, \frac{1}{2})$

$f'(-2) = \frac{-13}{(-2)^2} = -\frac{13}{4}$

Tangent line has Slope $m = -\frac{13}{4}$

$$f(x) = 7 + \frac{13}{x}$$



[End of Example 1]

[Example 2] (similar to Exercise 2.4#37) Let $f(x) = 7 + 13\sqrt{x}$

$\sqrt{x}$ type function

(A) Find $f'(x)$ using the ***Definition of the Derivative***.

(B) Find $f(1)$ and $f'(1)$.

(C) Find $f(4)$ and $f'(4)$.

(D) Find $f\left(\frac{1}{4}\right)$ and $f'\left(\frac{1}{4}\right)$.

(E) Find $f(0)$ and $f'(0)$.

(F) Illustrate your results from **(B),(C),(D),(E)** using a given graph of $f(x)$.

(A) We need to find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Step 1 find $f(x+h)$

$f(x) = 7 + 13\sqrt{x}$

$f(\quad) = 7 + 13\sqrt{(\quad)}$ empty version

$f(x+h) = 7 + 13\sqrt{(x+h)}$

$= 7 + 13\sqrt{x+h}$

Step 2 find $f(x+h) - f(x)$

$$f(x+h) - f(x) = \cancel{7} + 13\sqrt{x+h} - (\cancel{7} + 13\sqrt{x})$$ ← important parentheses!

Cancel 7 by arithmetic

$$= 13\sqrt{x+h} - 13\sqrt{x}$$

factor out 13 in front

$$= 13\left(\sqrt{x+h} - \sqrt{x}\right)$$

Trick: multiply and divide by a special term

$$= 13\left(\sqrt{x+h} - \sqrt{x}\right)\frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

multiply the numerator terms

$$= \frac{13\left(\sqrt{x+h}\sqrt{x+h} - \sqrt{x}\sqrt{x+h} + \sqrt{x+h}\sqrt{x} - \sqrt{x}\sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

Cancel green terms using arithmetic simplify red terms

$$= \frac{13(x + h - x)}{\sqrt{x+h} + \sqrt{x}} = \frac{13h}{\sqrt{x+h} + \sqrt{x}}$$

cancel using arithmetic

Step 3 Find $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\Big(f(x+h)-f(x)\Big)$$

$$= \frac{1}{h}\left(\frac{13h}{\sqrt{x+h}+\sqrt{x}}\right)$$

using result from step 2

Step 4 Find $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

Solution

$$f'(x) = \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{\text{indeterminate form}} = \lim_{h\to 0} \frac{1}{h}\left(\frac{13h}{\sqrt{x+h}+\sqrt{x}}\right)$$

use result from step 3

indeterminate form because $\frac{h}{h}$

Since $h \to 0$, we know $h \neq 0$, so we can cancel $\frac{h}{h}$

$$= \lim_{h\to 0} \frac{13}{\sqrt{x+h}+\sqrt{x}}$$

notice that this function is continuous at $h=0$, so we can find the limit by just substituting in $h=0$.

$$= \frac{13}{\sqrt{x+(0)}+\sqrt{x}} = \frac{13}{\sqrt{x}+\sqrt{x}} = \frac{13}{2\sqrt{x}}$$

We found that for $f(x) = 7 + 13\sqrt{x}$, the derivative is $f'(x) = \frac{13}{2\sqrt{x}}$

(B) find $f(1)$ and $f'(1)$

There was an incorrect y value in the video

Solution $f(1) = 7 + 13\sqrt{1} = 7 + 13 = 20$ ✓ point on graph at $(1, 20)$

$f'(1) = \frac{13}{2\sqrt{1}} = \frac{13}{2}$ tangent line has slope $m = \frac{13}{2}$

(C) find $f(4)$ and $f'(4)$

Solution: $f(4) = 7 + 13\sqrt{4} = 7 + 13\cdot(2) = 7 + 26 = 33$ point on graph at $(4, 33)$

$f'(4) = \frac{13}{2\sqrt{4}} = \frac{13}{2\cdot 2} = \frac{13}{4}$ tangent line slope is $m = \frac{13}{4}$

(D) find $f(1/4)$ and $f'(1/4)$

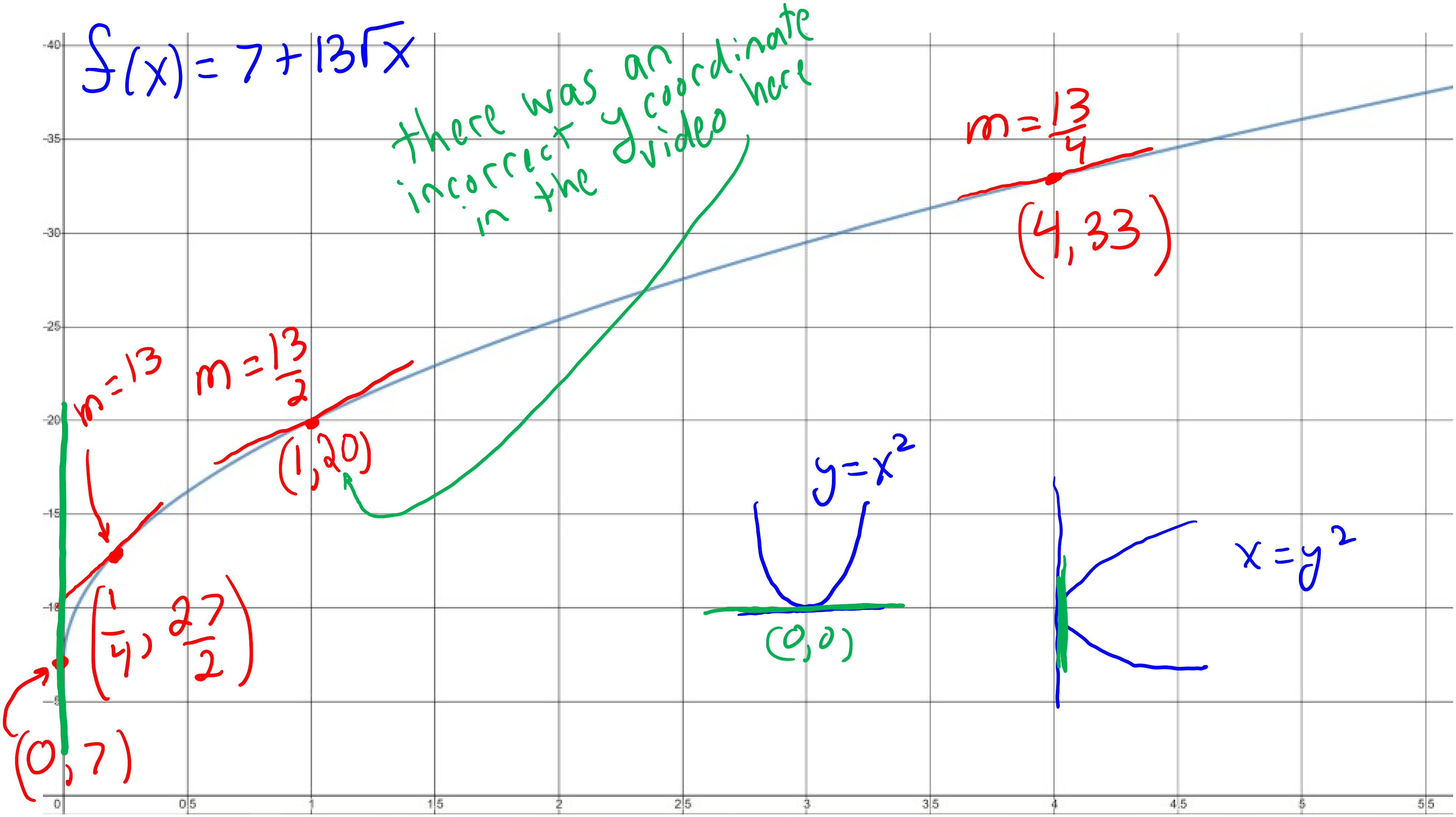
Solution: $f(1/4) = 7 + 13\sqrt{1/4} = 7 + 13\frac{1}{\sqrt{4}} = 7 + \frac{13}{2} = \frac{27}{2}$ Point at $\left(\frac{1}{4}, \frac{27}{2}\right)$

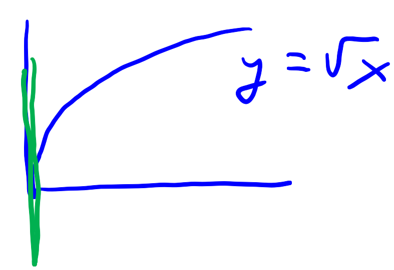
$f'(1/4) = \frac{13}{2\sqrt{1/4}} = \frac{13}{2(1/2)} = \frac{13}{1} = 13$ Tangent line slope is $m = 13$

(E) find $f(0)$ and $f'(0)$

Solution: $f(0) = 7 + 13\sqrt{0} = 7 + 13\cdot 0 = 7 + 0 = 7$ point at $(0, 7)$

$f'(0) = \frac{13}{2\sqrt{0}} = \frac{13}{2\cdot 0} = \frac{13}{0}$ does not exist!! $\boxed{???}$





[End of Example 2]

End of Video