Subject for this video: Tangent Line and Instantaneous Velocity

Reading:

- **General:** Section 2.5, Basic Differentiation Properties
- **More specifically:** middle of p. 151 – top of p. 152, Examples 7,8

Homework:

Tangent Line and Instantaneous Velocity (2.4#15,17) (2.5#59,63)

Background Concepts from Section 2.4, discussed in the Video for Homework H25

Definition of *Instantaneous Rate of Change*

words: the ***instantaneous rate of*** *change of* f *at* a

alternate words: the derivative of f at a

symbol: $f'(a)$

meaning: the number $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

remark: The instantaneous rate of change $f'(a)$ is a number.

Definition of the *Tangent Line*

words: *the line tangent to the graph of of* f *at* $x = a$

meaning: the line that has these two properties

- The line contains the point $(x, y) = \big(a, f(a)\big)$, which is called the ***point of tangency***.
- The line has slope $m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, called the ***tangent line slope***.

remark: the *tangent line slope* $m = f'(a)$ is also called the ***slope of the graph at*** $\boldsymbol{x = a}$**.**

The point slope form of the equation for the line tangent to the graph of $\boldsymbol{f(x)}$ at $\boldsymbol{x = a}$

$$\big(y - f(a)\big) = f'(a) \cdot (x - a)$$

Terminology of *Position* and *Velocity*

Time: When our book uses mathematical functions to describe the motion of objects, x is a variable that represents the elapsed time.

Position: To say an object is *moving in 1 dimension* means that it can go forward or backward in one direction but cannot turn. In such situations, a single coordinate can be used to keep track of the position of the object. A function called the *position function* gives the value of the coordinate at a given time. In our book, the position function is called f. That is, at time x, the coordinate of the object is the number $f(x)$.

average velocity: The words *average velocity from time $x = a$ to time $x = b$* mean the same thing as *average rate of change of position from time $x = a$ to time $x = b$*. That is, the number

$$m = \frac{f(b) - f(a)}{b - a}$$

instantaneous velocity: The words *instantaneous velocity at time $x = a$* mean the same thing as *instantaneous rate of change of position at time $x = a$*. That is, the number

$$m = f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Notice that the computation of an *instantaneous rate of change*, or a *velocity*, or of a *tangent line slope*, involves finding a *derivative*. When we discussed those three kinds of quantities back in Section 2.4, the only way that we knew how to compute those derivatives was to use the *Definition of the Derivative*, with our computations organized by the outline of the *Four Step Proccess*.

Now that we are in Section 2.5, we can compute deriviatives more easily, using the *Derivative Rules*. This will make problems about *instantaneous rate of change* and *tangent lines* and *velocity* much easier.

[Example 1] (Similar to 2.4#15)

For $f(x) = \sqrt{x^2 + 9}$, the *slope of the graph* is known to be $\frac{4}{5}$ at the point where $x = 4$.

Find the equation for the line tangent to the graph of $f(x)$ at that point.

Remark: The given information could have been given in a different form.

It could have said that the *instantaneous rate of change* is known to be $\frac{4}{5}$ at $x = 4$.

Solution We are being asked to find the equation for the tangent line.

$$(y - f(a)) = f'(a)(x - a)$$

We need to find the three quantities a, $f(a)$, $f'(a)$.

$a = 4$ = given x coordinate of the point of tangency

$f'(4) = \frac{4}{5}$ the known slope of the graph at the point where $x = 4$

So the tangent line slope is $m = f'(4) = \frac{4}{5}$

We need to figure out $f(a)$

$f(a) = f(4) = \sqrt{(4)^2 + 9} = \sqrt{16+9} = \sqrt{25} = 5$

this is the y coordinate of the point of tangency

Now we can build the equation for the tangent line

$$y - \underbrace{f(a)}_{f(a)=5} = \underbrace{f'(a)}_{f'(a)=\frac{4}{5}}(x - \underset{a=4}{a})$$

$$(y - 5) = \frac{4}{5}(x - 4)$$

Equation for the tangent line in point slope form

Convert to slope intercept form by solving for y

$$y - 5 = \frac{4}{5}(x - 4)$$

$$= \frac{4}{5}x - \frac{16}{5}$$

$$y = \frac{4}{5}x - \frac{16}{5} + 5$$

$$y = \frac{4}{5}x + \frac{9}{5}$$

Equation for the tangent line

[Example 2] (Similar to 2.5#59) Let $f(x) = x^3 - 3x^2 - 9x + 11$

(a) Find $f'(x)$.

(b) Find the slope of the line that is tangent to the graph of f at $x = 3$.

(c) Find the slope of the line that is tangent to the graph of f at $x = 0$.

(d) Find the x coordinates of all points on the graph of f that have horizontal tangent lines.

(e) Find the equation of the line that is tangent to the graph of f at $x = 2$. Show all details clearly and present your equation in slope intercept form.

(f) Illustrate your results for (b),(c),(d),(e) on a given graph of $f(x)$.

Solution

(a) $f'(x) = \frac{d}{dx}\left(x^3 - \underline{3}x^2 - \underline{9}x + \underline{11}\right)$

use the sum and constant multiple rule

$$= \underbrace{\frac{d}{dx}x^3 - 3\frac{d}{dx}x^2 - 9\frac{d}{dx}x}_{\text{use power rule}} + \frac{d}{dx}\underline{11}$$

constant function rule

$n=3$ $\quad n=2$ $\quad n=1$

$$= 3\cdot x^{3-1} - 3(2x^{2-1}) - 9(1\cdot x^{1-1}) + 0$$

$$= 3x^2 - 6x - 9$$

(b) We are asked to find the slope of the line tangent to the graph at $x=3$.

This means that we need to find $m = f'(3)$

This means to substitute $x=3$ into the formula for $f'(x)$

We found $f'(x) = 3x^2 - 6x - 9$

so $m = f'(3) = 3(3)^2 - 6(3) - 9$

$= 3 \cdot 9 - 18 - 9$

$= 27 - 27$

$= 0$

(c) To find the slope of the tangent line at $x=0$, we should compute $m = f'(0)$

result: $m = f'(0) = 3(0)^2 - 6(0) - 9$

$= -9$

(d) Find x coordinates of all points that have horizontal tangent lines.

Solution: Horizontal lines have slope $m=0$.

we want $m = 0 = f'(x)$

So set $f'(x)=0$ and solve for x.

$$0 = f'(x)$$
$$= 3x^2 - 6x - 9$$

factor out a 3

$$= 3(x^2 - 2x - 3)$$

factor some more

$$= 3(x+1)(x-3)$$

We see that $f'(x)$ will be zero when $x=-1$ or $x=3$.

those are the x values where the graph will have a horizontal tangent line.

(e) Find the equation of the line tangent to the graph at $x=2$.

Solution: We need to build the equation $(y-f(a)) = f'(a)\cdot(x-a)$

✓ Get parts

$a=2$ (the x coordinate of the point of tangency)

$f(a) = f(2) = (2)^3 - 3(2)^2 - 9(2) + 11 = 8 - 12 - 18 + 11 = 19 - 30 = -11$

sub $x=2$ into $f(x) = x^3 - 3x^2 - 9x + 11$

This number is the y coordinate of the point of tangency

$f'(a) = f'(2) = 3(2)^2 - 6(2) - 9 = 12 - 12 - 9 = -9$

Sub $x=2$ into $f'(x) = 3x^2 - 6x - 9$

This number is the slope of the tangent line

✓ Substitute the parts into the equation

$(y - (-11)) = -9\cdot(x - 2)$

Tangent Line Equation Point Slope form

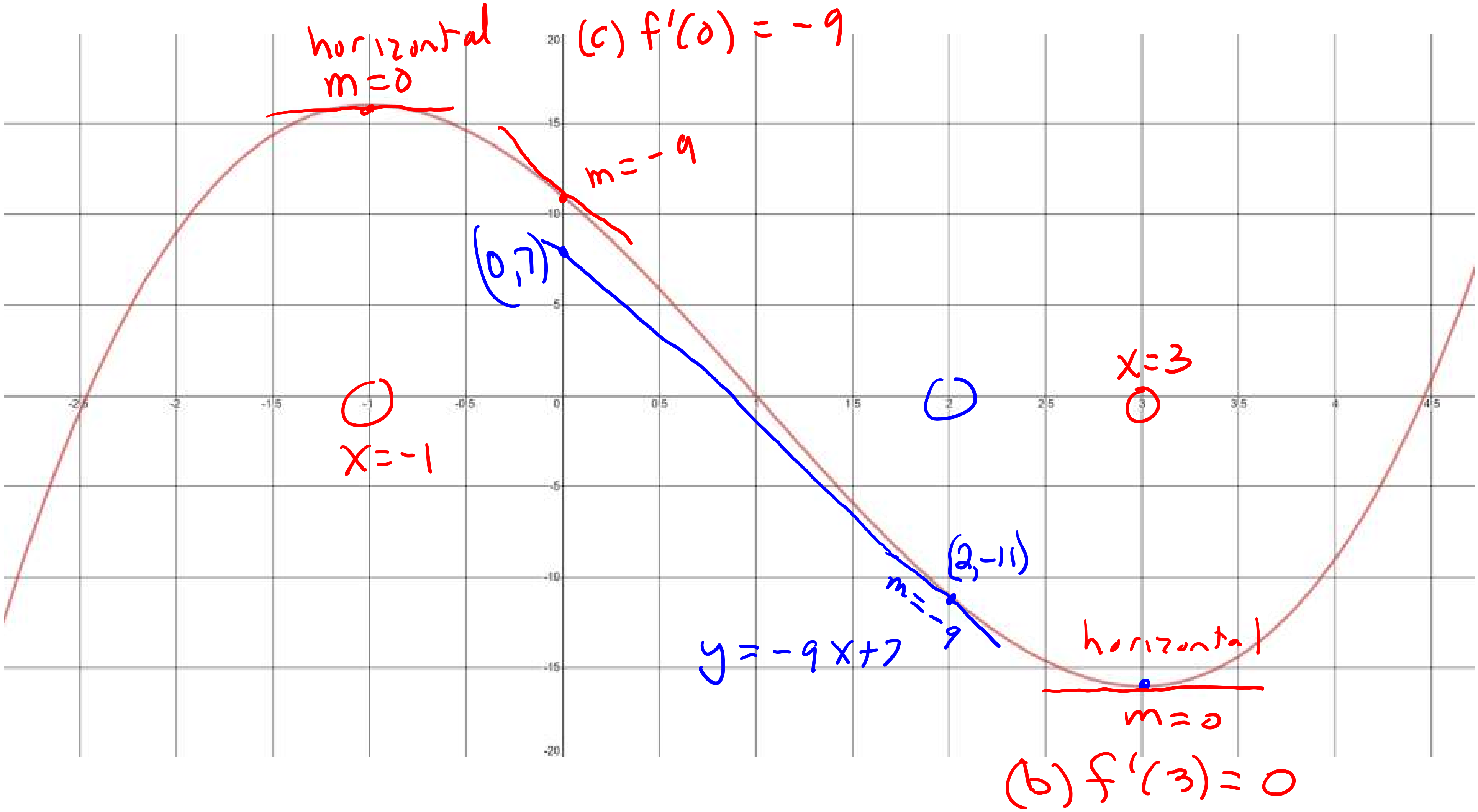
Convert to slope intercept form by solving for y

$y + 11 = -9x + 18$

$y = -9x + 7$

Equation for the tangent line in slope intercept form

Given graph of $f(x) = x^3 - 3x^2 - 9x + 11$



End of [Example 2]

[Example 3] (Similar to 2.5#63)

An object is moving along a straight track with position function $f(x) = 2x^3 - 21x^2 + 60x$, where x is the time in seconds and $f(x)$ is the position in meters at time x.

(a) Find the *velocity function*.

(b) Find the *velocity* at time $x = 3$

(c) ~~(b)~~ Find the *velocity* at time $x = 0$

(d) ~~(c)~~ At what times is the velocity $v = 0$?

(e) ~~(d)~~ Illustrate your results for (b),(c) on a given graph of $f(x)$.

Solution

(a) velocity function $v(x) = f'(x)$

$$= \frac{d}{dx}(2x^3 - 21x^2 + 60x)$$

$$= 2\frac{d}{dx}x^3 - 21\frac{d}{dx}x^2 + 60\frac{d}{dx}x \quad n=3 \quad n=2 \quad n=1$$

$$= 2(3 \cdot x^{3-1}) - 21(2 \cdot x^{2-1}) + 60(1 \cdot x^{1-1})$$

$$= 6x^2 - 42x + 60$$

(b) to find the velocity at time $x=3$, we must find $V(3)$.

$$v(3) = f'(3) = 6(3)^2 - 42(3) + 60 = 54 - 126 + 60 = -12$$

Substitute $x=3$ into $f'(x) = 6x^2 - 42x + 60$

(c) To find the velocity at time $x=0$, we must find $v(0)$

$$v(0) = f'(0) = 6(0)^2 - 42(0) + 60 = 60$$

sub $x=0$ into $f'(x) = 6x^2 - 42x + 60$

(d) To find when the velocity is equal to zero, we must set the velocity equal to zero and solve for x.

$$0 = v(x) = f'(x) = 6x^2 - 42x + 60$$

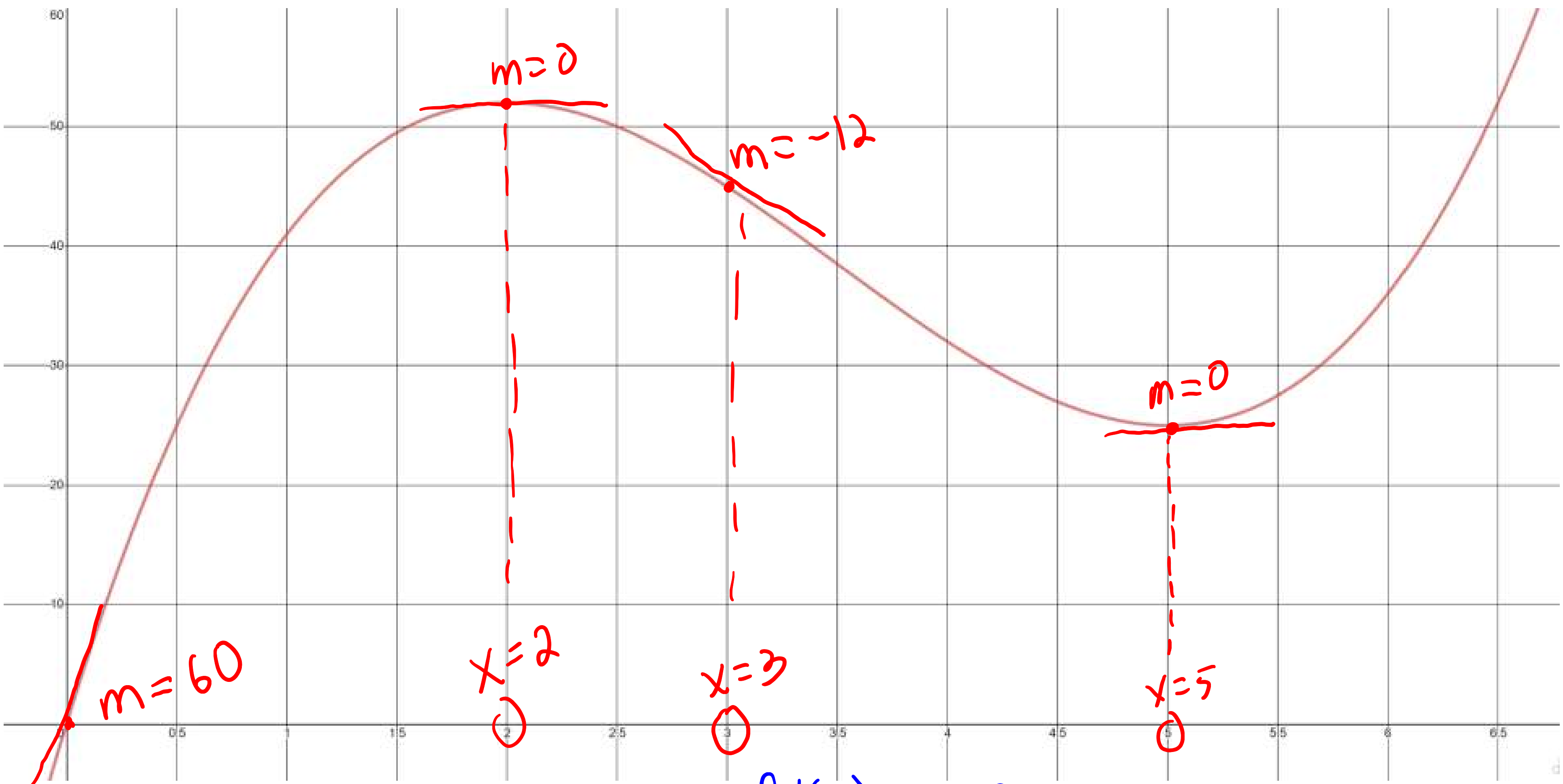
Solve by factoring

$$0 = 6(x^2 - 7x + 10) = 6(x-2)(x-5)$$

Solutions are $x=2$ and $x=5$

So those are the times when the velocity is zero

(e) Given graph of $f(x) = 2x^3 - 21x^2 + 60x$ = position at time x.



(b) $v(3) = f'(3) = -12$

(c) $v(0) = f'(0) = 60$

(d) $v(2) = f'(2) = 0$ and $v(5) = f'(5) = 0$

[End of Example]

End of Video