Subject for this video: Using Marginal Quantities to Estimate Change in Quantities

Reading:

- **General:** Section 2.7, Marginal Analysis
- **More Specifically:** Middle of p. 162 – bottom of p.167, parts of Examples 1,2,3

Homework:

H36: Using Marginal Quantities to Estimate Change in Quantities (2.7#33,43,45)

Exact Change

In the video for Homework H34, we discussed the idea of *exact change* in a quantity.

Suppose that a company manufactures some item, and the cost of producing a batch of x of the items is given by a cost function $C(x)$.

Suppose that x_0 is a particular value of the variable x. For instance, x_0 could be $x_0 = 50$.

And suppose that the company increases the number of items made from x_0 items to $x_0 + 1$ items.

Question: What will be the resulting change in the cost?

Question: What will be the resulting change in the cost?

Answer: The change in cost is $\Delta C = C(x_0 + 1) - C(x_0)$

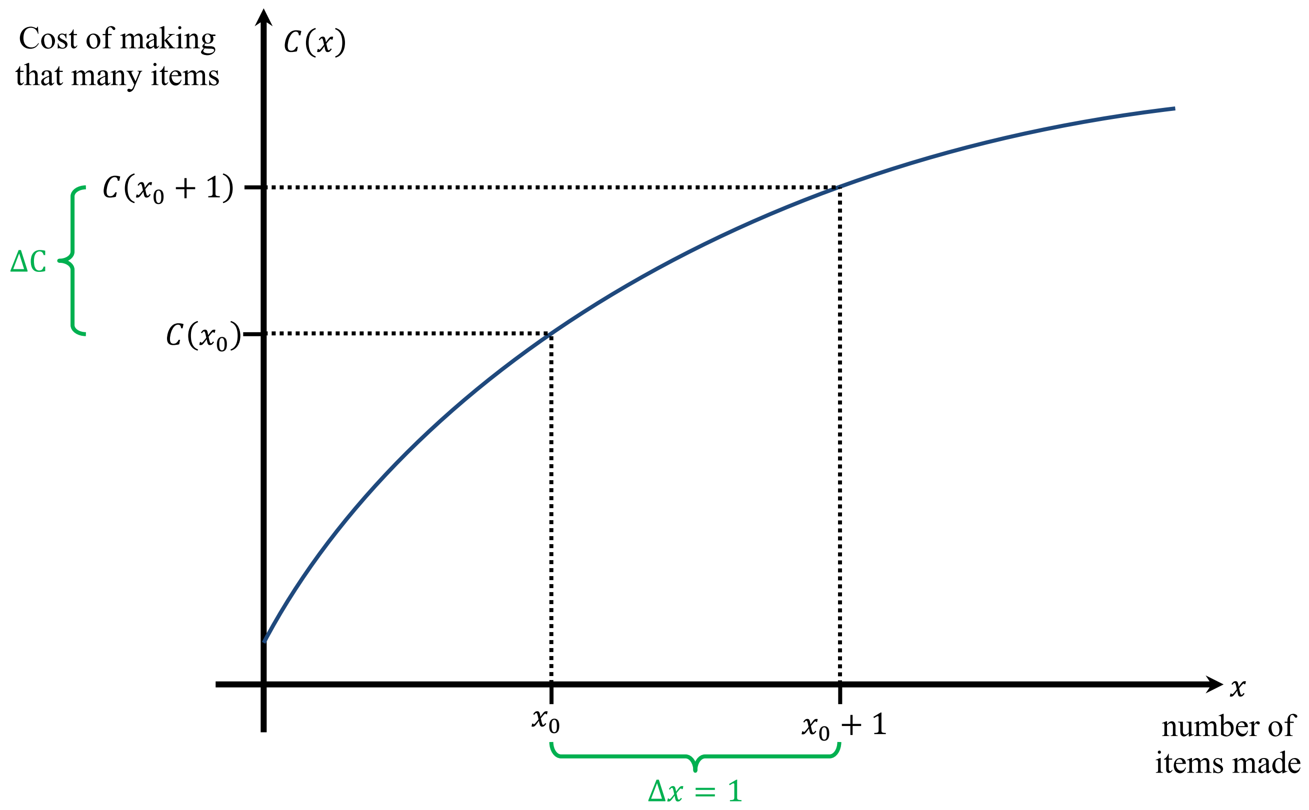
Remarks:

- This is called the *exact change in cost*.
- The book calls this *the exact cost of the* $(x_0 + 1)^{st}$ *item*.
- The change in quantity would be $\Delta x = (x_0 + 1) - x_0 = 1$

For example, if a company increases the number of items made from 50 to 51 items,

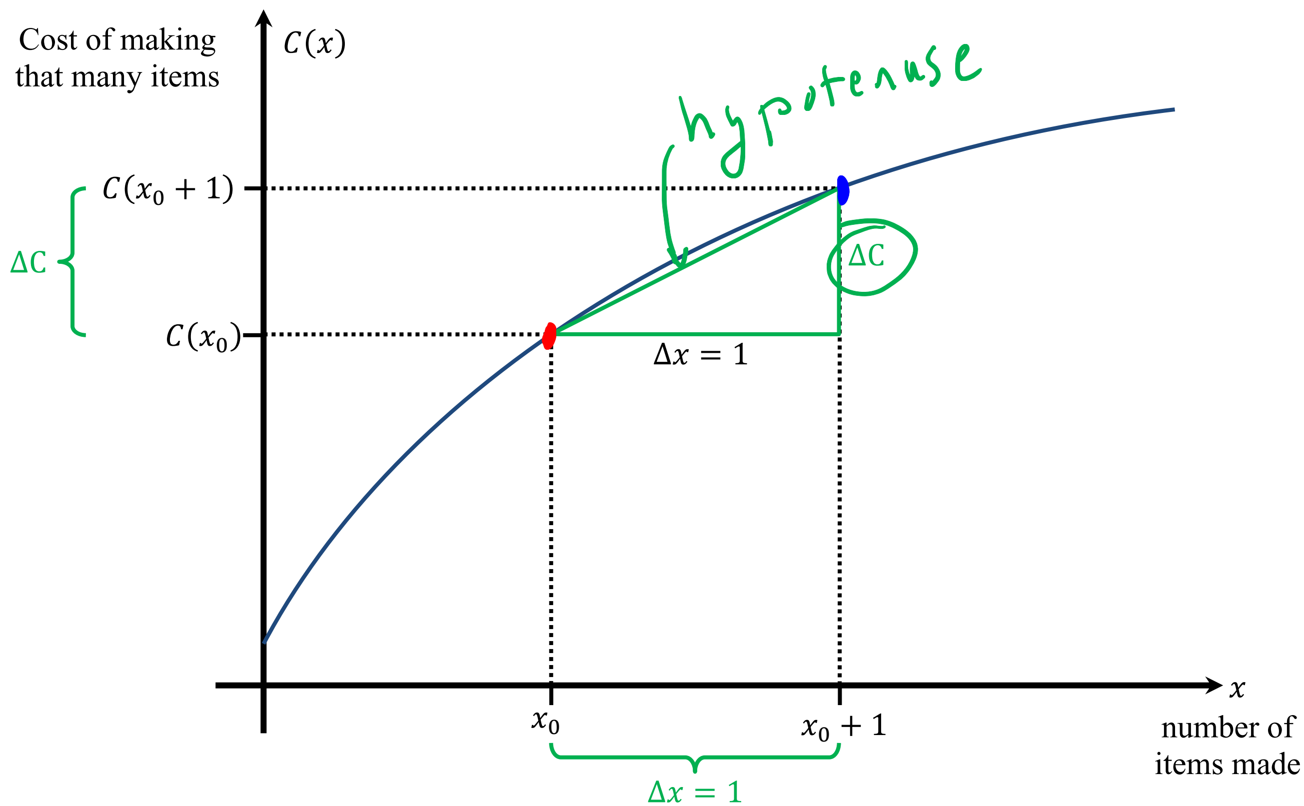
- the *exact change in cost* is $\Delta C = C(51) - C(50)$
- The book would call this the *exact cost of producing the 51st item*.

These quantities could be illustrated on the graph of a cost function as shown.



.

Let's add some features to this graph. A green right triangle could be drawn as shown.



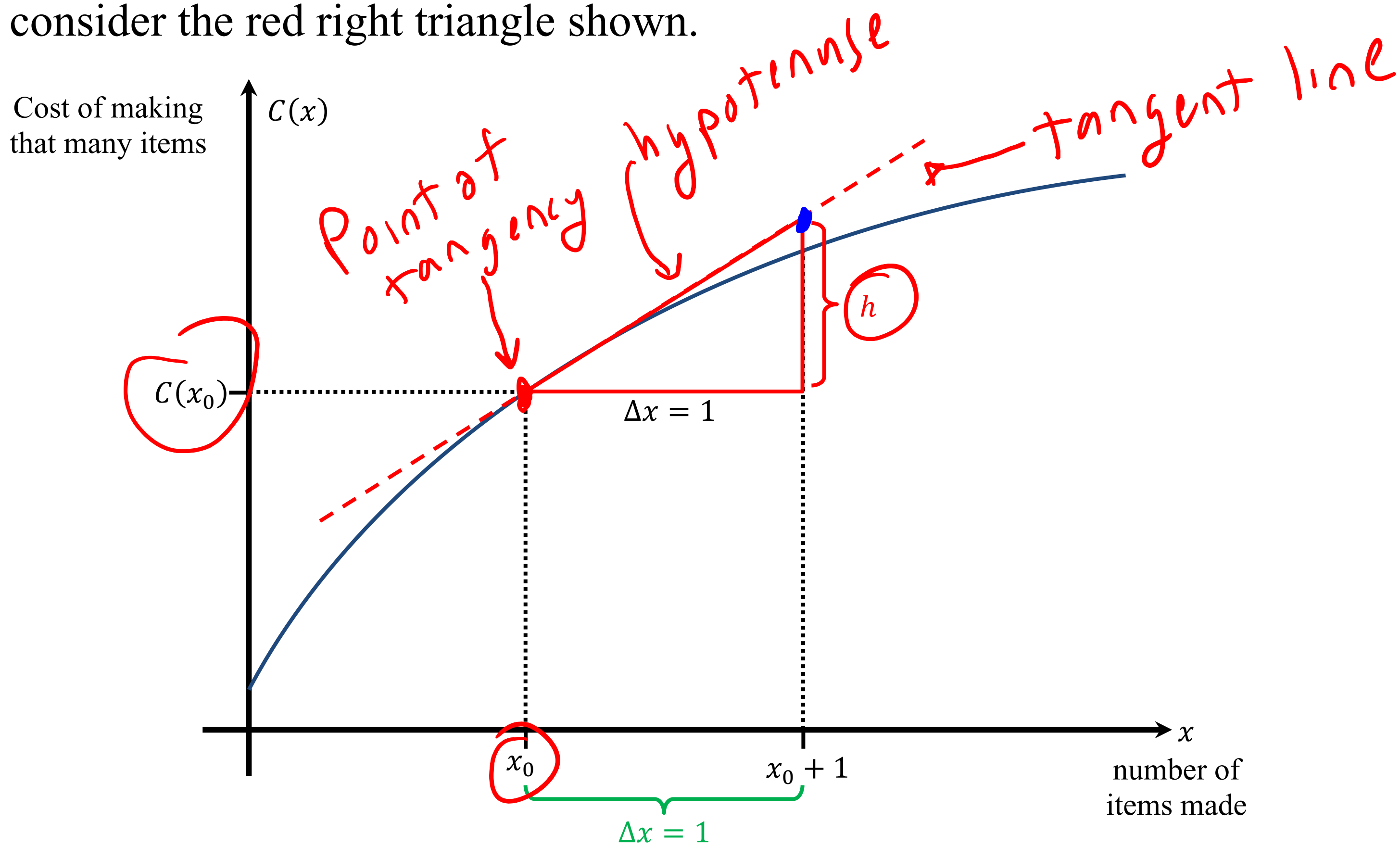
In this green triangle,

- The hypotenuse of the triangle is part of a *secant line* for $C(x)$.
  - One end of the hypotenuse is at the point $(x_0, C(x_0))$ on the graph of $C(x)$
  - The other end of the hypotenuse is at the point $(x_0 + 1, C(x_0 + 1))$ on the graph of $C(x)$
- The height of the triangle is the exact change in cost, $\Delta C = C(x_0 + 1) - C(x_0)$.

.

Approximate Change

Now consider the red right triangle shown.

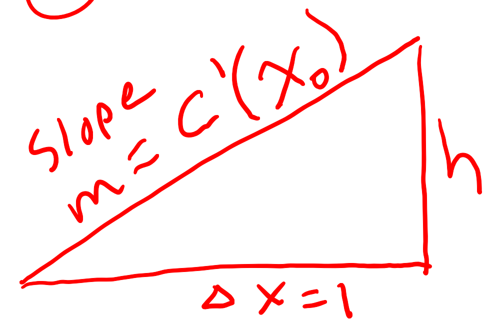


In this red triangle,

- The hypotenuse of the triangle is part of a *tangent line* for $C(x)$.
  - One end of the hypotenuse is at the point $(x_0, C(x_0))$ on the graph of $C(x)$
  - The other end of the hypotenuse is at a point on the tangent line, not on the graph of $C(x)$
- The height of the triangle is unknown. For now, we'll just call the height h..

Now consider this question: What would be the value of the height h?

To answer this, note that the height h is part of a right triangle



The triangle has base $\Delta x = 1$.

The hypotenuse of the triangle is part of the line tangent to the graph of $C(x)$ at $x = x_0$.

Therefore, the hypotenuse slope is $m = C'(x_0)$.

But

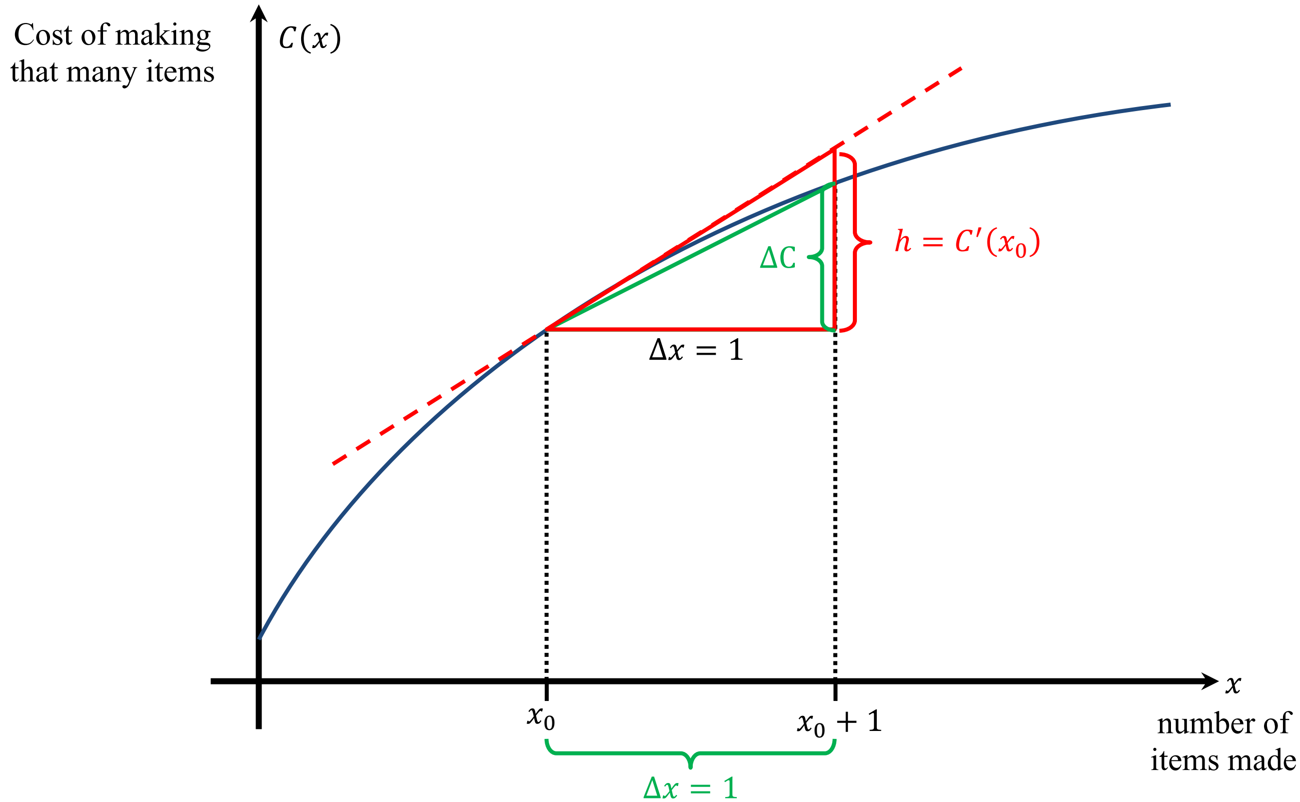
$$slope = \frac{rise}{run}$$

That is,

$$C'(x_0) = \frac{h}{1} = h$$

We have determined that $h = C'(x_0)$. Let's add that information to our drawing of the red triangle.

For comparison, let's also show the green triangle.



Notice that the green height ΔC is not the same as the red height $h = C'(x_0)$, but they are close. For that reason, we can think of the quantity $h = C'(x_0)$ as the *approximate change in cost*.

To reiterate

- Exact Change in cost is $\Delta C = C(x_0 + 1) - C(x_0)$.
- Approximate change in cost is $C'(x_0)$.

The two are not the same, but they are close:

$$\text{Exact Change} \approx \text{Approximate Change}$$

$$\underbrace{\Delta C}_{\substack{\text{Exact}\\ \text{Change}}} = \underbrace{C(x_0 + 1) - C(x_0)}_{\substack{\text{Exact}\\ \text{Change}}} \approx \underbrace{C'(x_0)}_{\substack{\text{Approximate}\\ \text{Change}}}$$

The ideas just discussed for a cost function can be generalized to other functions.

Remember that the quantity $C'(x_0)$ is called the marginal cost at a production level of x_0.)

In general, for other functions, the idea is

Using Marginal Quantities to Approximate Change in Quantities

$$\text{Exact Change} \approx \text{Approximate Change}$$

$$\underbrace{\Delta Q}_{\substack{\text{Exact}\\ \text{Change}}} = \underbrace{Q(x_0 + 1) - Q(x_0)}_{\substack{\text{Exact}\\ \text{Change}}} \approx \underbrace{Q'(x_0)}_{\substack{\text{Approximate}\\ \text{Change}}}$$

.

[Example 1] (Similar to 2.7#33) (Note that parts (A),(B),(C) of this example are similar to problems 2.7#4,5,6, and were done in the video for Homework H34)

The total cost of producing x electric guitars is $C(x) = 1000 + 100x - 0.25x^2$ dollars.

(A) What is the cost of producing a batch of 50 guitars?

Solution

$C(50) = 1000 + 100(50) - .25(50)^2 = \cdots = \$5375$

from video for H34

(B) What is the cost of producing a batch of 51 guitars?

Solution

$C(51) = 1000 + 100(51) - .25(51)^2 = \cdots = \$5449^{\underline{75}}$

from video for H34

(C) If batch size changes from $x = 50$ guitars to $x = 51$ guitars, what will be change in the cost of producing a batch of guitars? That is, what is ΔC? (exact value)

(The book calls this quantity *the cost of producing the 51st guitar*)

Solution

$\Delta C = C(51) - C(50) = \$5449^{\underline{75}} - \$5375 = \$74^{\underline{75}}$

exact change in cost

(D) If the batch size changes from $x = 50$ guitars to $x = 51$ guitars, use the marginal cost function to find an approximate value for the change in the cost of producing a batch of guitars. That is, use the marginal cost function to find an approximation for ΔC.

(Book wording: *Use marginal cost to approximate the cost of producing the 51*[st] *guitar.*)

Solution The approximate change in cost will be $C'(50)$

Strategy: Find $C'(x)$, then substitute in $x=50$ to get $C'(50)$

$$C'(x) = \frac{d}{dx}\left(1000 + 100x - .25x^2\right)$$

use sum and constant multiple rule

$$= \frac{d}{dx}1000 + 100\frac{d}{dx}x - .25\frac{d}{dx}x^2$$

use constant function rule and power rule

$$= 0 + 100(1 \cdot x^{1-1}) - .25(2 \cdot x^{2-1})$$

$$= 100x^0 - .5x = 100 - .5x$$

$$C'(50) = 100 - .5(50) = 100 - 25 = 75$$

approximate change in cost

Business Terminology

In the previous videos, we discussed *Demand*, *Cost*, *Revenue*, and *Profit*.

And we discussed the idea of *Marginal Quantities*.

Business Terminology

Demand, x (small letter), is a variable that represents the number of items made. This sounds simple enough, but there can be complications. For example, in some problems, x represents the number of thousands of items made.

Cost, $C(x)$ (capital letter C), is a function that gives the cost of making the batch of x items.

Revenue, R (capital letter), is the amount of money that comes in from the sale of the x items that are made.

Profit, $P(x)$ (capital letter P), is a function defined as follows

$$Profit = Revenue - Cost$$

$$P(x) = R(x) - C(x)$$

The expression ***Marginal Quantity*** means *The Derivative of Quantity*. That is, **Marginal Revenue** is $R'(x)$, and **Marginal Cost** is $C'(x)$, and **Marginal Profit** is $P'(x)$.

.

More Business Terminology

To that terminology from previous videos, we will today add one more new term: *price*

Price, p (small letter), is a variable that represents the selling price per item.

The **Price Demand Equation** is just what it says: an equation that relates the price p and the Demand x. For example $2x + 3p = 10$ could be a Price Demand Equation.

In some situations, the Price Demand Equation can be solved for p in terms of x. For example, when the equation above is solved for p, the result is $p = -\frac{2}{3}x + \frac{10}{3}$. Notice that the resulting equation describes price p as a *function* of Demand x. We could use function notation to indicate this, writing $p(x) = -\frac{2}{3}x + \frac{10}{3}$. This is called the **price function**.

Revenue, R (capital letter), is the amount of money that comes in from the sale of the x items that are made. Because of our simplifying assumptions listed above, we can say that

$$Revenue = (number\ of\ items\ sold) \cdot (selling\ price\ per\ item)$$
$$Revenue = Demand \cdot Price$$
$$R(x) = x \cdot p(x)$$

.

[Example 2] (Similar to 2.7#43) A company makes hats.

Price p (in dollars) and *Demand* x for hats are related by the equation $x = 1000 - 20p$.

(A) Find *price* p in terms of x. The result is the *price function* $p(x)$.

Find the domain of the price function.

Solution:

$$x = 1000 - 20p$$

$$x - 1000 = -20p$$

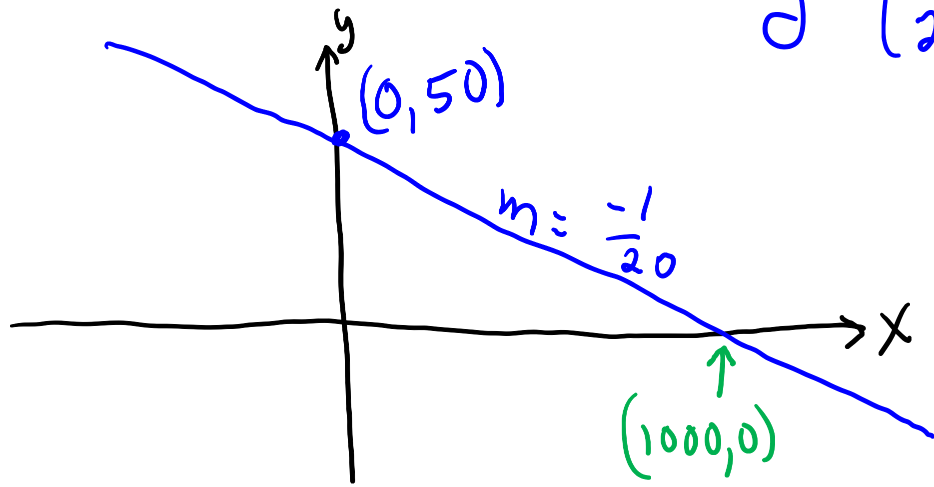
$$-x + 1000 = 20p$$

$$\left(-\frac{1}{20}\right)x + \frac{1000}{20} = p$$

$$p(x) = \left(-\frac{1}{20}\right)x + 50$$

graph the price function $p(x) = \left(\frac{-1}{20}\right)x + 50$

First graph the abstract equation $y = \left(\frac{-1}{20}\right)x + 50$



Find x intercept by setting $y=0$ and solving for x

$$0 = \left(\frac{-1}{20}\right)x + 50$$

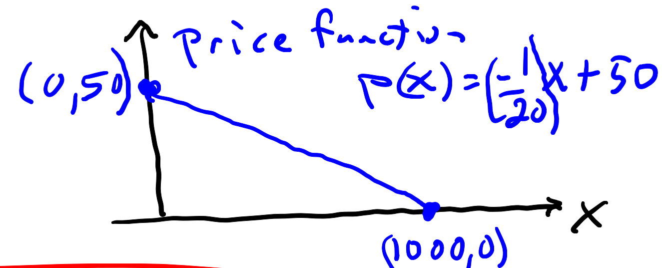
$$\left(\frac{1}{20}\right)x = 50$$

$$x = 50 \cdot 20 = 1000$$

So the abstract equation $y = \left(\frac{-1}{20}\right)x + 50$ has domain all real numbers,

Cant make a negative number of items, So for the price function, must have $x \geq 0$

Also, the selling price $p(x)$ cannot be negative. So must have $x \leq 1000$



So the domain for the price function is $0 \leq x \leq 1000$

(B) Find the *revenue* $R(x)$ from the sale of x hats. What is the domain of $R(x)$?

Solution: Revenue = Demand · price

$$R(x) = x \cdot p(x)$$

$$= x\left(\left(-\frac{1}{20}\right)x + 50\right)$$

$$R(x) = \left(-\frac{1}{20}\right)x^2 + 50x$$

The domain of the Revenue function will be

$$0 \le x \le 1000$$

price function is only valid for

$$0 \le x \le 1000$$

(C) Find the *marginal revenue at a production level of* 400 *hats* and interpret the results.

Solution: We are being asked to find $R'(400)$

Strategy: Find $R'(x)$, then substitute in $x = 400$ to get $R'(400)$

$$R'(x) = \frac{d}{dx}\left(\left(-\frac{1}{20}\right)x^2 + 50x\right) = \left(-\frac{1}{20}\right)\frac{d}{dx}x^2 + 50\left(\frac{d}{dx}x\right)$$

$$= \left(-\frac{1}{20}\right)\left(2 \cdot x^{2-1}\right) + 50\left(1 \cdot x^{1-1}\right) = \left(-\frac{1}{10}\right)x + 50$$

Now substitute in $x = 400$

$$R'(400) = \left(-\frac{1}{10}\right)(400) + 50 = -40 + 50 = 10$$

To interpret this result, we must explain what it tells us about selling hats.

- The revenue from selling a batch of 401 hats would be roughly \$10 more than the revenue from selling 400 hats.

$$\Delta R = R(401) - R(400) \approx R'(400) = 10$$

- At a production level of 400 hats, Revenue will increase at a rate roughly \$10 per hat.

(D) Find the marginal revenue at a production level of 650 hats and ***interpret the results.***

Solution We must find $R'(650)$

$R'(650) = \left(-\frac{1}{10}\right)(650) + 50 = -65 + 50 = -15$

sub $x = 650$ into $R'(x)$

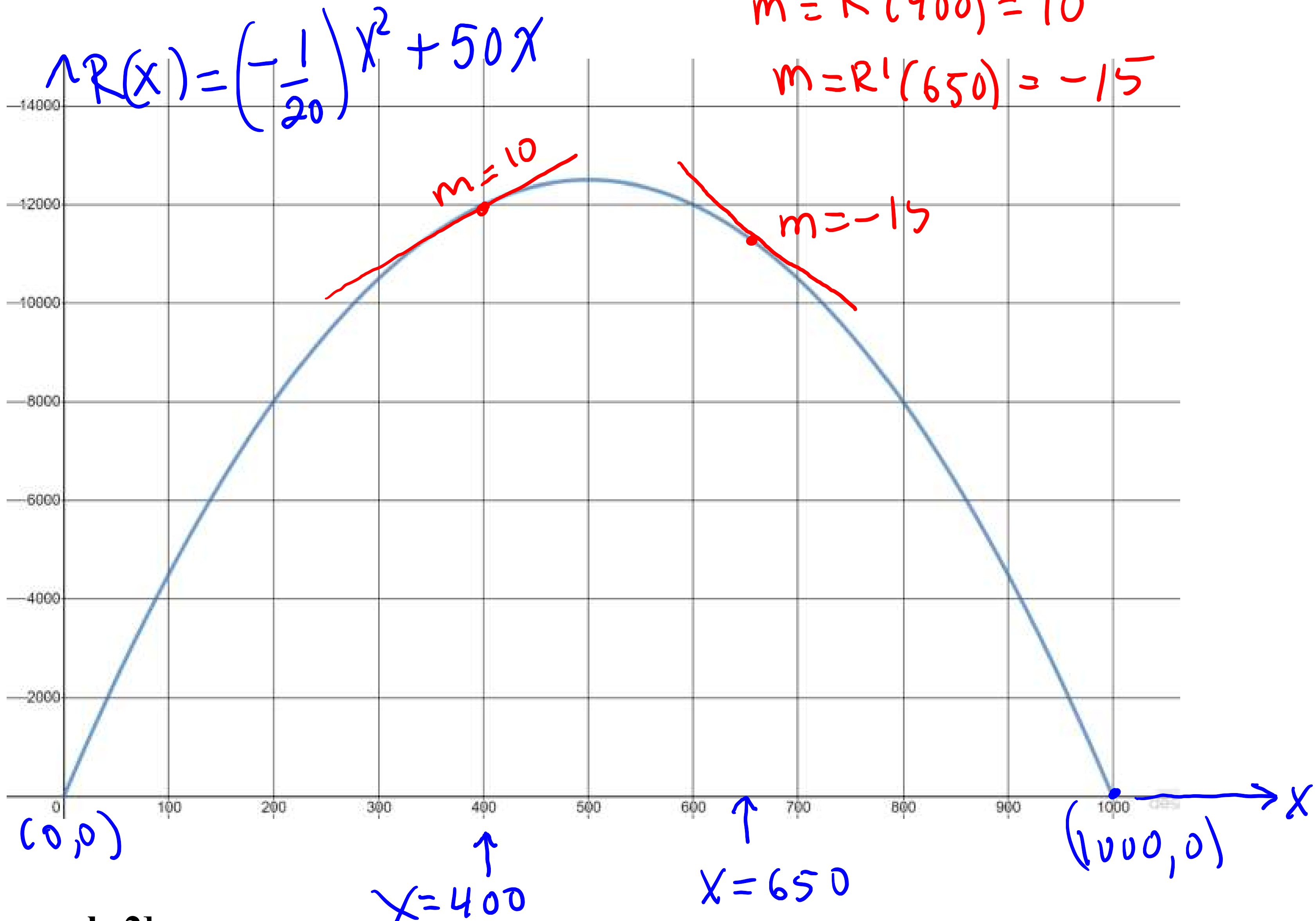
Interpretations

- The Revenue from selling a batch of 651 hats would be roughly \$15 less than the revenue from selling a batch of 650 hat
- At a production level of 650 hats, the revenue will decrease at a rate of roughly \$15 per hat.

(E) Illustrate the results of **(C)**,**(D)** on this given graph of $R(x)$.

$m = R'(400) = 10$

$m = R'(650) = -15$



End of [Example 2]

End of Video