Subject for this video: The Constant e and the Function $e^{(x)}$

Reading:

- **General:** Section 3.1 The Constant e and Continuous Compound Interest
- **More Specifically:** Page 181

Homework:

H37: The Constant e and the Function $e^{(x)}$ (3.1#21,23)

The constant *e*

Consider the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Do we have any intuition about whether this limit exists, and what might be its value?

Notice two things about the terms in the limit.

(1) The thing in the parentheses will always be greater than 1, because $\frac{1}{n}$ is a positive number.

That thing in parentheses is being raised to higher and higher powers, because $n \to \infty$.

We know that when a number greater than 1 gets raised to higher and higher powers, the result gets more and more positive, without bound. For instance

$$2^3 = 8,\ \ 2^4 = 16,\ \ 2^5 = 32,\ \ 2^6 = 64,\ \ 2^7 = 128, ...$$

Based on this, we might expect that the limit above could maybe be

$$maybe\ \ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \infty$$

(2) But the thing in parentheses is getting closer and closer to the value 1, because $\frac{1}{n}$ is getting closer and closer to 0. We know that 1 raised to any power is just 1.

$$1^3 = 1, \;\; 1^4 = 1, \;\; 1^5 = 1, \;\; 1^6 = 1, \;\; 1^7 = 1, \dots$$

Based on this observation, we might expect that the limit above could maybe be

$$maybe \quad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 1$$

So it would be believable if the limit were ∞, and it would also be belivable if the limit were 1.

Or perhaps the limit turns out to not be ∞ and to not be 1, but some other real number.

Let's explore values of $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$, to see if we can see a pattern emerge.

Values of $\left(1+\frac{1}{n}\right)^n$ as $n \to \infty$

n	$\left(1+\frac{1}{n}\right)^n$
10	$\left(1+\frac{1}{10}\right)^{10} \approx 2.594$
100	$\left(1+\frac{1}{100}\right)^{100} \approx 2.705$
1000	2.717
10,000	2.718
$\vdots$	$\vdots$
$n \to \infty$	a number near 2.718

Based on this table, we might guess that The limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and its value is a number near 2.718

BIG FACTS FROM HIGHER MATH

- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ *does* exist.
- The symbol e is used to denote the real number that is the value of the limit. That is

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

- The number e is between 2 and 3 That is, $2 < e < 3$
- The value of e is near 2.718 but e is irrational. (The number e cannot be written exactly as a fraction or as a terminating decimal, or even as a repeating decimal.)
- The number e is called *Euler's number*.

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Next, consider this related limit:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$$

It turns out (by doing change of variables) that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

This is a *function of* x, called the *natural exponential function*. I like to put use parentheses around the x in the formulas for functions, so I will rewrite it that way and give it an offical introduction.

The ***natural exponential function*** is defined by the equation

$$e^{(x)} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$$

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Graphs of Exponential Functions

The surprising way that we arrived at the function $y = e^{(x)}$ can be set aside for now, and we can turn our attention to discussing the *behavior* of $y = e^{(x)}$ and of other other exponential functions.

When b is a real number such that $0 < b < 1$ or $1 < b$, the function $b^{(x)}$ is called the ***exponential function with base b***, or the ***base b exponential function***

The most common exponential functions are $e^{(x)}, 2^{(x)}, 10^{(x)}$

It will help our understanding of exponential functions if we review how their graphs are constructed and then study the resulting graphs. We will build graphs for $2^x, 3^x, e^x$.

Values of $2^{(x)}, 3^{(x)}, e^{(x)}$

$2 < e < 3$

x	$2^{(x)}$	$e^{(x)}$	$3^{(x)}$
-3	$2^{(-3)} = \frac{1}{2^3} = \frac{1}{8}$	$e^{(-3)} = \frac{1}{e^3}$	$3^{(-3)} = \frac{1}{3^{(3)}} = \frac{1}{27}$
-2	$2^{(-2)} = \frac{1}{2^2} = \frac{1}{4}$	$e^{(-2)} = \frac{1}{e^2}$	$3^{(-2)} = \frac{1}{3^2} = \frac{1}{9}$
-1	$2^{(-1)} = \frac{1}{2^{(1)}} = \frac{1}{2}$	$e^{(-1)} = \frac{1}{e}$	$3^{(-1)} = \frac{1}{3^1} = \frac{1}{3}$
0	$2^{(0)} = 1$	$e^{(0)} = 1$	$3^{(0)} = 1$
1	$2^{(1)} = 2$	$e^{(1)} = e$	$3^{(1)} = 3$
2	$2^{(2)} = 4$	$e^{(2)} = e^2$	$3^{(2)} = 9$
3	$2^{(3)} = 8$	$e^{(3)} = e^3$	$3^{(3)} = 27$

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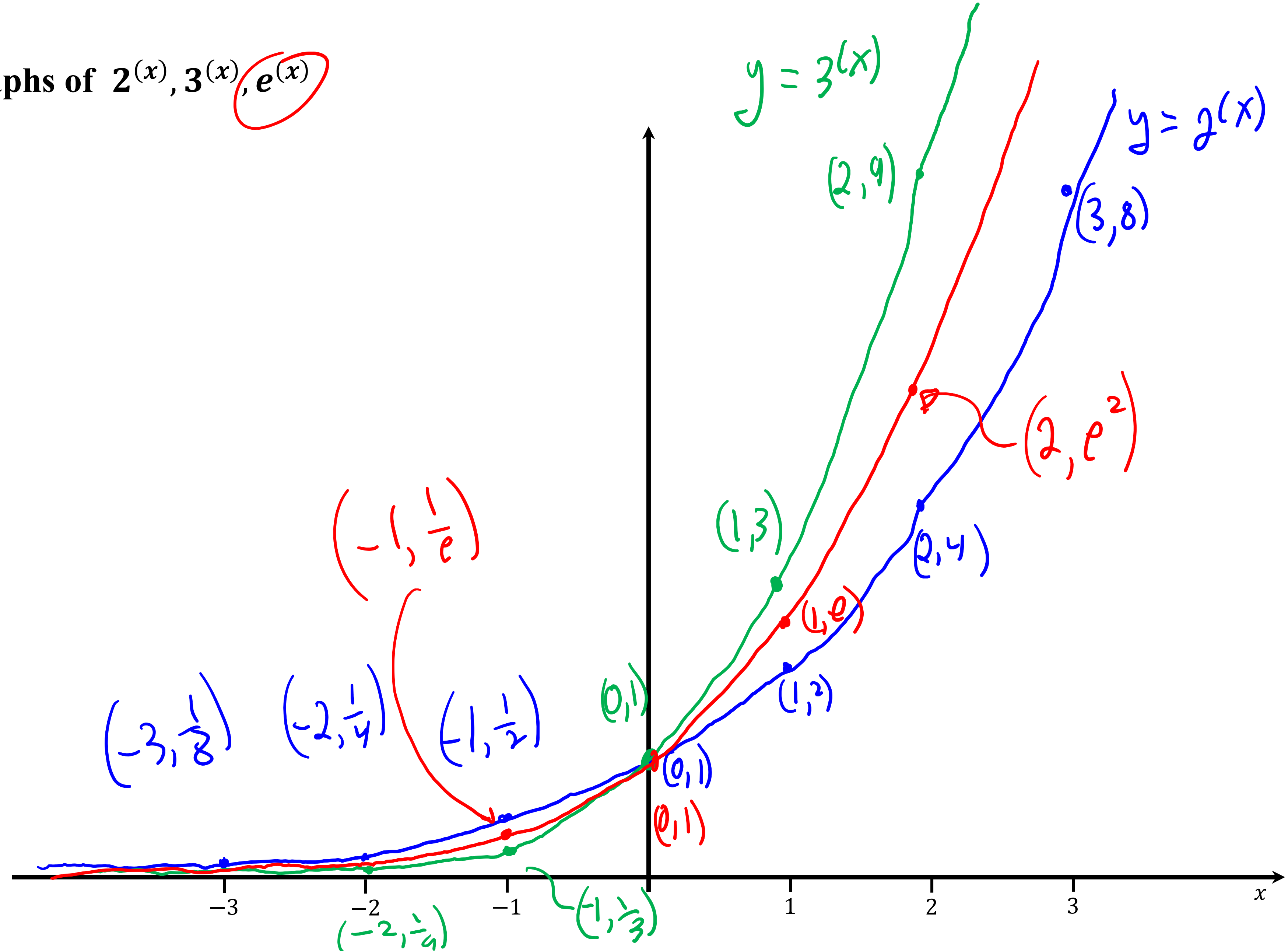
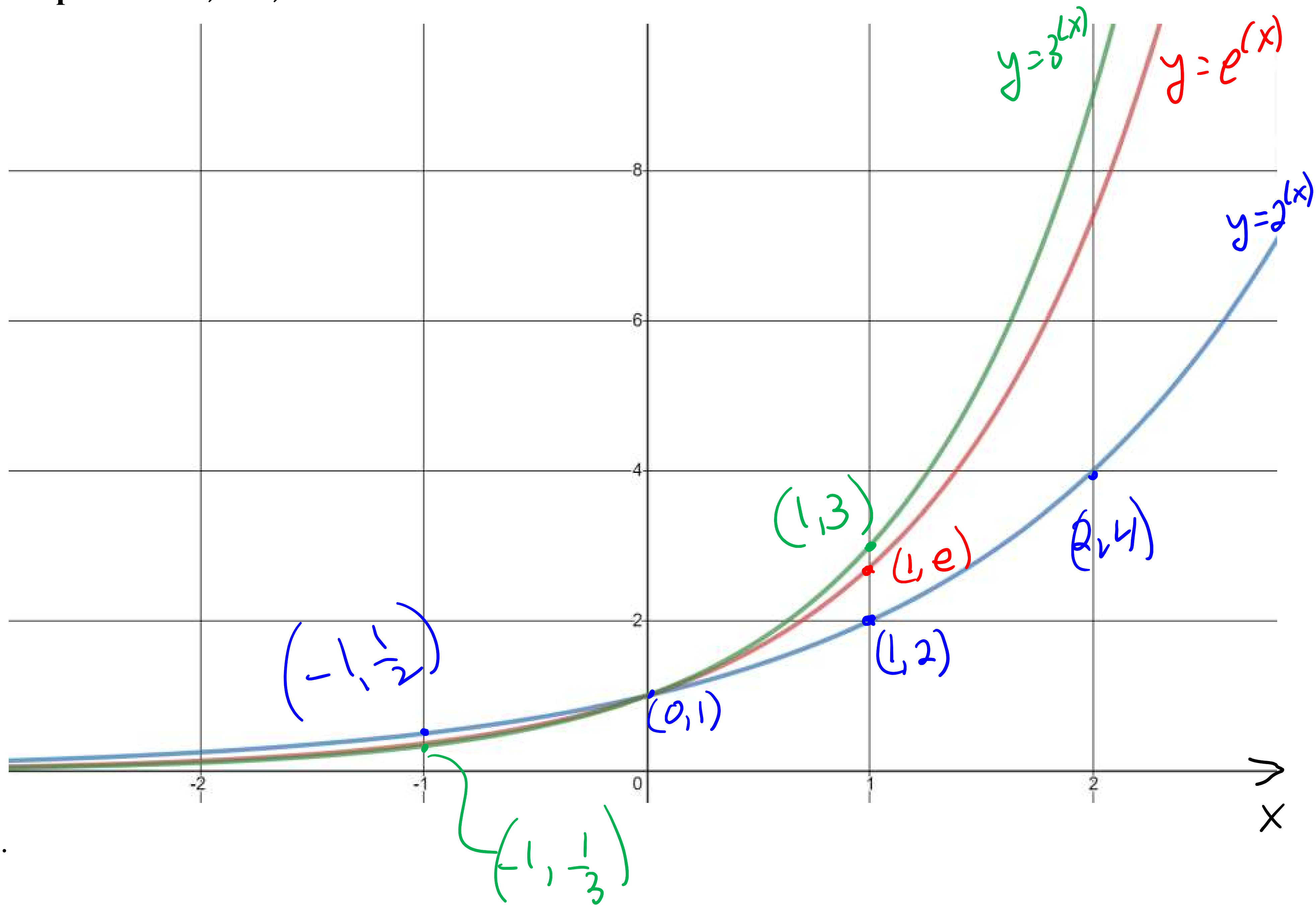
rule of exponents $b^{-p} = \frac{1}{b^p}$

Graphs of $2^{(x)}, 3^{(x)}, e^{(x)}$



The hand-drawn graph is not meant to be to scale. Rather it is just meant to show the relationship between the three graphs. On the next page, we will consider a computer-drawn graph of the functions and then discuss general characteristics.

Graphs of $2^{(x)}, 3^{(x)}, e^{(x)}$



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Properties of Exponential Functions $b^{(x)}$ with $b > 1$

- Domain and Range
  - The domain is the set of all real numbers x. In interval notation, $(-\infty, \infty)$
  - The range is all $y > 0$. In interval notation, $(0, \infty)$
- The graph has three distinctive points:
  - The graph goes through the point $(x, y) = (0,1)$ because $b^{(0)} = 1$
  - The graph goes through the point $(x, y) = (1, b)$ because $b^{(1)} = b$
  - The graph goes through the point $(x, y) = \left(-1, \frac{1}{b}\right)$ because $b^{(-1)} = \frac{1}{b}$
- End Behavior
  - The graph goes up without bound on the right. That is, $\lim_{x\to\infty} b^{(x)} = \infty$
  - Graph has a horizontal asymptote on left with equation $y = 0$. That is, $\lim_{x\to-\infty} b^{(x)} = 0$
- The graph is increasing from left to right. That is, if $x_1 < x_2$, then $b^{(x_1)} < b^{(x_2)}$

End of Video

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