Subject for this video: Continuously-Compounded Interest

Reading:

- **General:** Section 3.1 The Constant e and Continuous Compound Interest
- **More Specifically:** Bottom of page 181 – Top of p.185, Examples 1,3,4

Homework:

H38: Continuously-Compounded Interest (3.1#11,27,29,31,35,37)

In this video, we will explore ways of computing interest on a bank account. We begin with the most basic type, *Simple Interest*, in which interest is only earned on the amount of the original deposit. We then consider the more complicated type called *Periodically-Compounded Interest*, in which the earned interest is periodically added to the original deposit, and then interest is earned on this larger pot of money, a process called *compounding*. Imagining the trend as the compounding happens more and more frequently leads us to the idea of *Continuously-Compounded Interest*. It's all very concrete, real-world stuff. But along the way, an amazing thing happens: we see the emergence, in the formulas, of the famous mathematical constant e, called *Euler's number*.

Simple Interest

To say that a bank account has ***Simple Interest*** means that the interest is computed only on the amount of the original deposit.

Simple Interest Formula

An account with ***simple interest*** has an account balance described by the equation

$$A = P + Prt = P(1 + rt)$$

In this equation,

P is the amount of the original deposit, called the *principal*

r is the *interest rate*, expressed as a decimal

t is the *time* in years since the original deposit

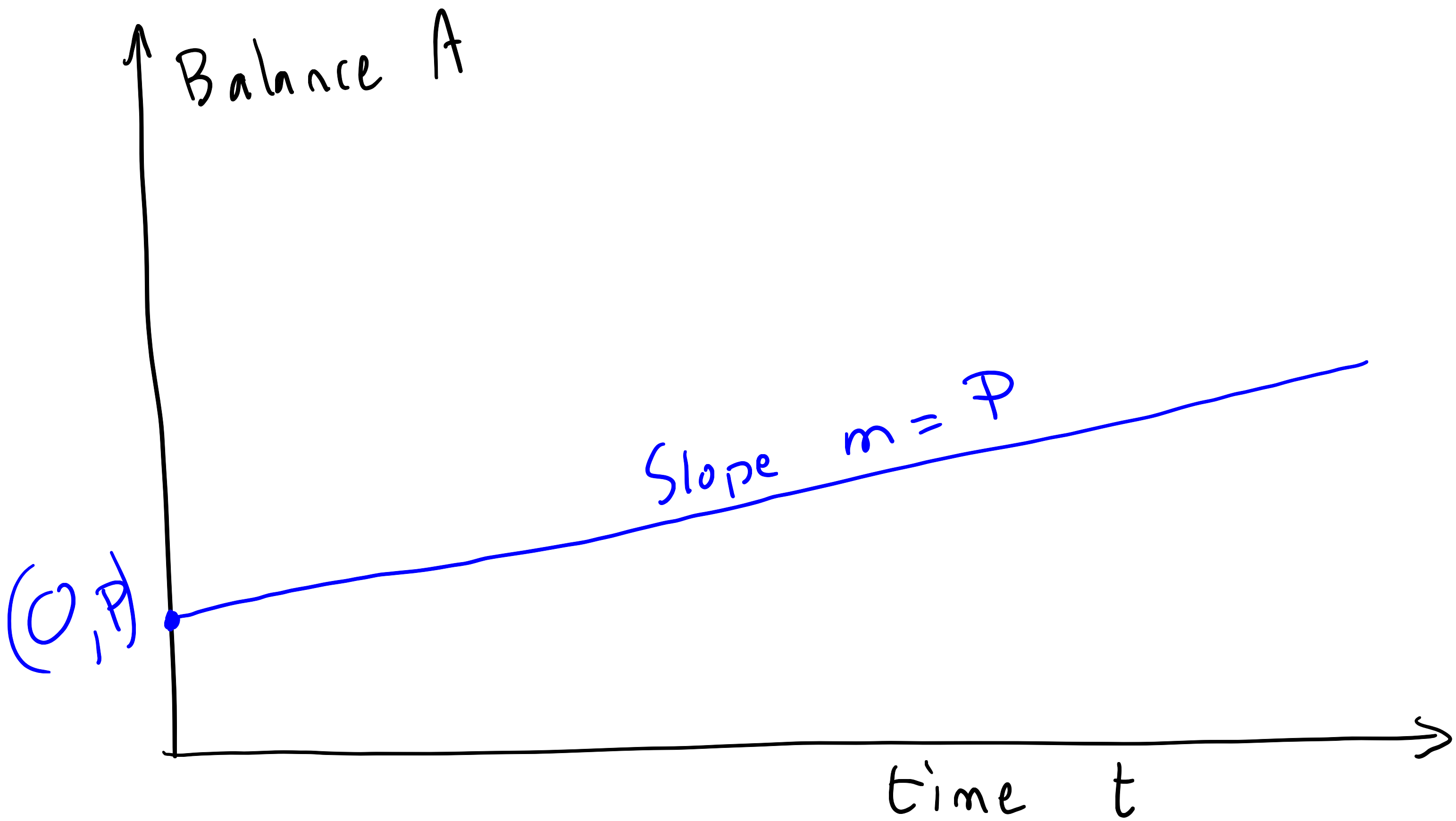
A is the *account balance* at time t.

Remark: The equation above is solved for A in terms of P, r, t. So it describes the balance A as a *function* of P, r, t. Using function notation, we would write

$$A(t) = P + Prt = P(1 + rt)$$

.

Notice that the formula $A(t) = P + Prt$ has the form of a line equation $y = mx + b$ with slope $m = Pr$ and y intercept at the point $(0, P)$.



[Example 1] Deposit \$1000 into bank account with 3% simple interest.

(A) What will the balance be after 7 years?

(B) Illustrate your result of *A* with a graph.

Solution $A = P + Prt = P(1 + rt)$

$P = 1000$

$r = .03$

$t = 7$

$A =$ unknown find A

$A = 1000(1 + .03(7)) = 1000(1 + .21) = 1000(1.21)$

$= \$1210$

Periodically Compounded Interest

For a bank account that has ***Periodically-Compounded Interest***, the interest is initially computed only on the amount of the original deposit, so the account initially behaves just like an account with a *simple interest*. But after a certain time interval, the earned interest is added to the principal, and the interest is then earned on the larger pot of money. This is called *compounding the interest*. The compounding happens repeatedly, at regular intervals.

Periodically-Compounded Interest

An account with ***periodically-compounded interest*** has a balance described by the equation

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

In this equation,

P is the amount of the original deposit, called the *principal.*

r is the *interest rate*, expressed as a decimal.

t is the *time* in years since the original deposit.

m is the number of times per year that the interest is *compounded*.

$A(t)$ is the *account balance* at time t.

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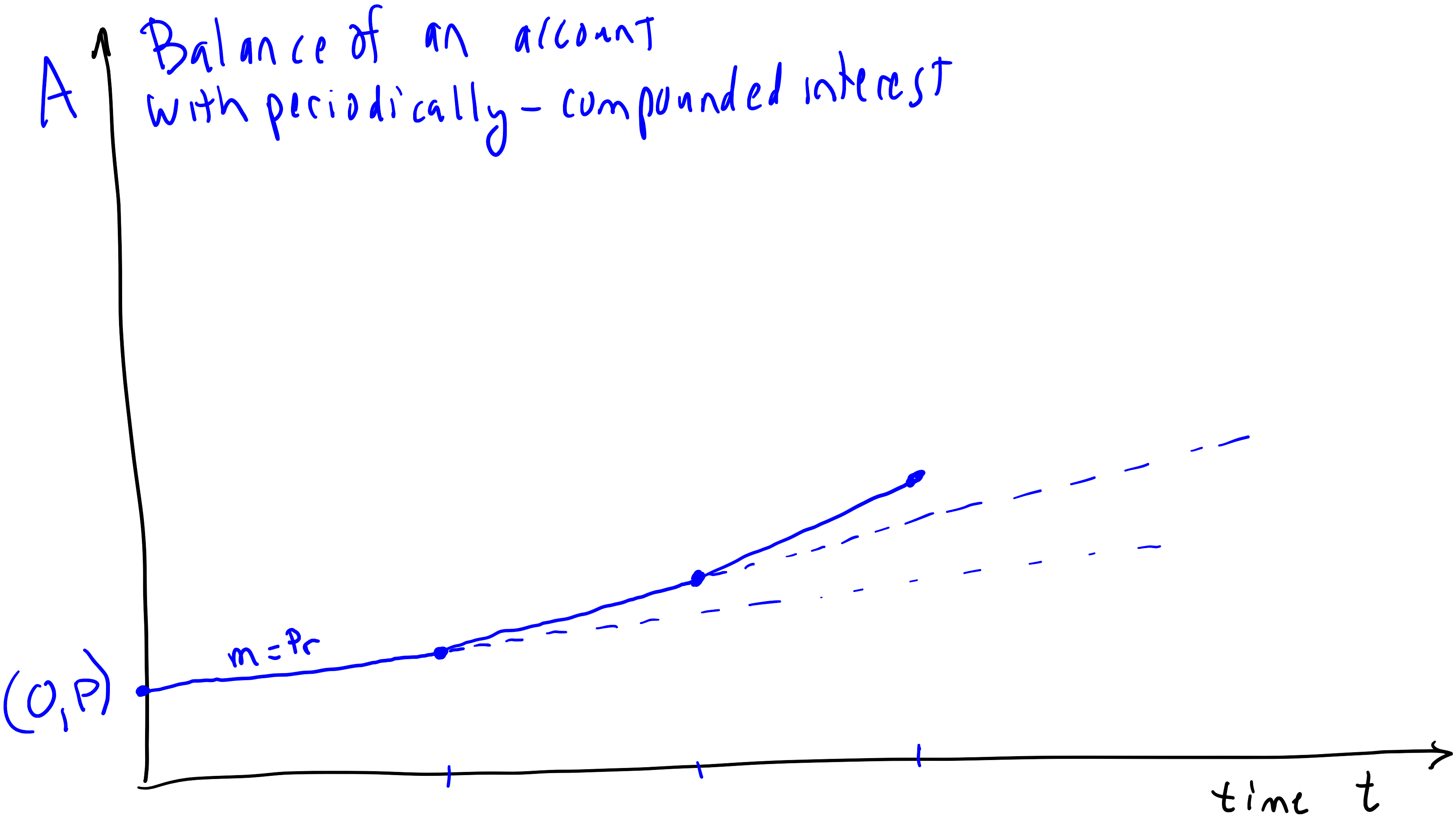
Actually, the formula $A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$ can only be used to compute the interest at the exact times when the interest is being compounded. In between those times, the interest is being earned as simple interest on a certain amount of money, so the account balance will grow like the balance on an account with simple interest. The graph of the account balance will be a segmented graph.



[Example 2] \$1000 is deposited into a bank account with 3% interest *compounded periodically*.

What will be the balance after 7 years,

(A) if the interest is *compounded yearly*? $m=1$

(B) if the interest is *compounded monthly*? $m=12$

(C) if the interest is *compounded daily*? $m=365$

(Give exact answers in symbols, then decimal approximations rounded to two decimal places.)

Solution $P=1000$ $r=.03$ $t=7$ A = unknown, find A

(A) $A = \underbrace{1000\left(1+\frac{.03}{1}\right)^{1\cdot 7}}_{\text{exact answer in symbols}} \approx \underbrace{\$1229^{\underline{87}}}_{\text{decimal approximation}}$

from Wolfram Alpha

(B) $A = \underbrace{1000\left(1+\frac{.03}{12}\right)^{12\cdot 7}}_{\text{exact answer in symbols}} \approx \$1233^{\underline{35}}$

from Wolfram Alpha

(C) $A = \underbrace{1000\left(1+\frac{.03}{365}\right)^{365\cdot 7}}_{\text{exact answer in symbols}} \approx \$1233^{\underline{67}}$

from Wolfram Alpha

Observation:

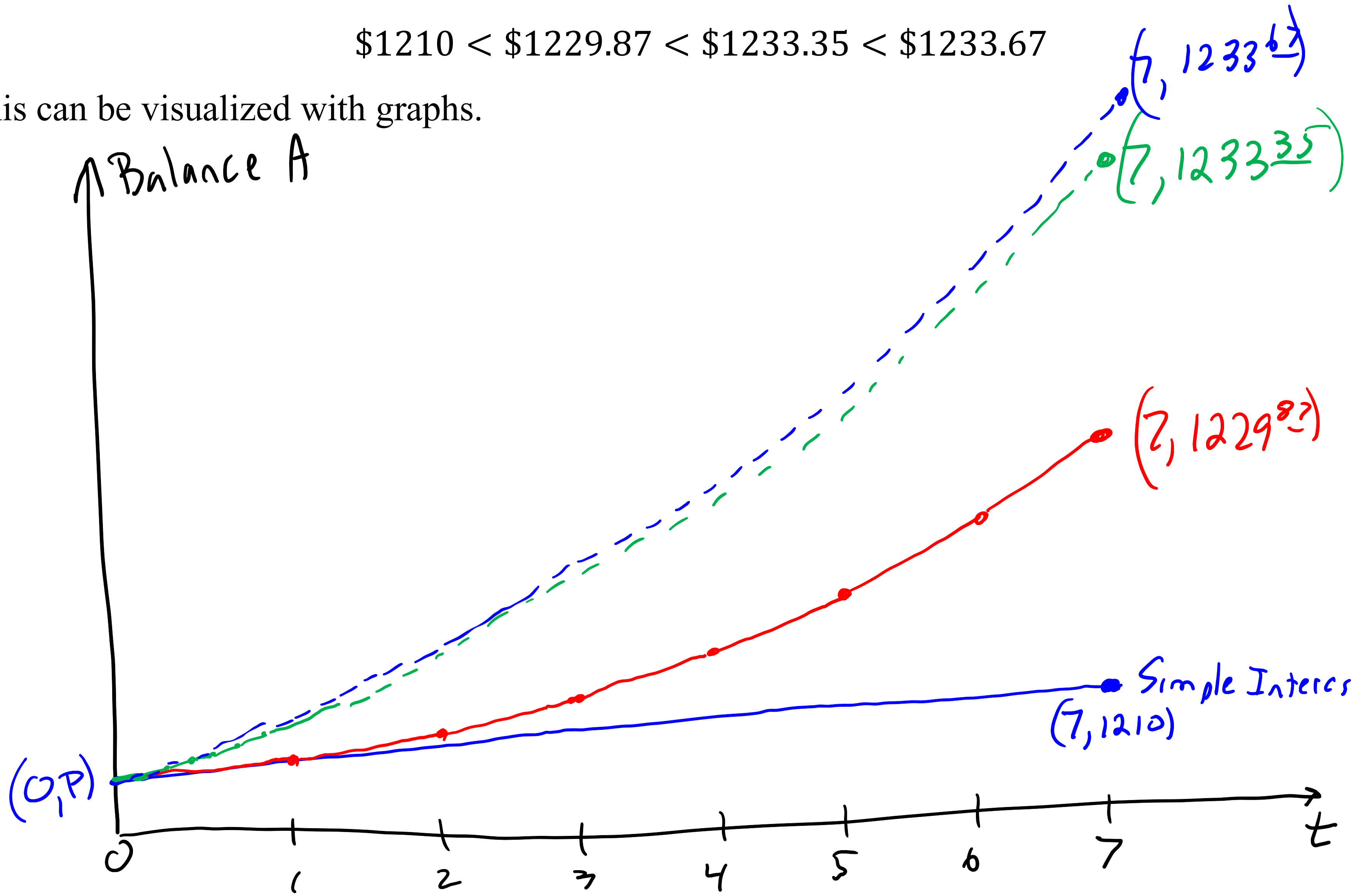
In the results of **[Example 1]** and **[Example 2]**, we observe that for more frequent compounding

$$\text{never compounded, then } m = 1,\ m = 12,\ m = 365$$

we get larger balances at 7 years:

$$\$1210 < \$1229.87 < \$1233.35 < \$1233.67$$

This can be visualized with graphs.



Obvious Question: What will happen to the value of $A = P\left(1+\frac{r}{m}\right)^{m\cdot t}$ as $m \to \infty$?

That is, what is the value of the limit

$$\lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{m\cdot t}$$

In the video for Homework H37, we discussed that a similar-looking limit has a famous value

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}} = e \approx 2.718$$

And we discussed that with an x inside, the limit can be figured out by doing a change of variable. The result of the limit is an exponential function.

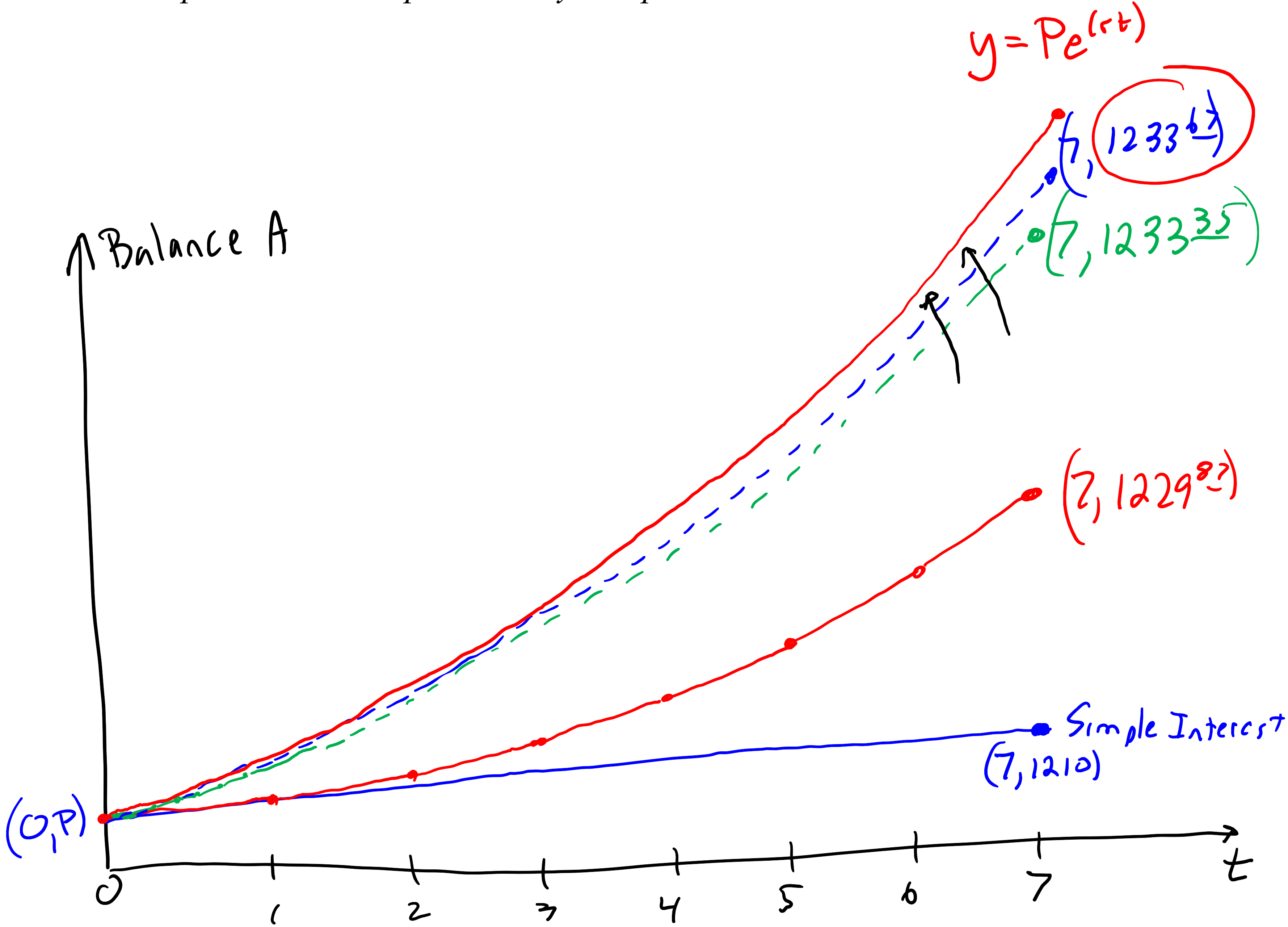
$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}} = e^{(x)}$$

It turns out (by doing another change of variables) that

$$\boxed{\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}} = \boxed{Pe^{rt}}$$

exponential function

Consider the graph of $y = Pe^{(rt)}$ in relation to the graphs of the functions describing the balance of accounts with *simple interest* and *periodically-compounded interest*.



Inspired by this, *invent* a new kind of bank account that uses the formula $A = Pe^{(rt)}$ to compute its balance. Call it *continuously compounded interest*.

Continuously-Compounded Interest Formula

An account with ***continuously-compounded interest*** has a balance described by the equation

$$A = Pe^{(rt)}$$

In this equation,

P is the amount of the original deposit, called the *principal*.

r is the *interest rate*, expressed as a decimal.

t is the *time* in years since the original deposit.

A is the *account balance* at time t.

[Example 3] (Similar to 3.1#27A)

\$1000 is deposited into a bank account with 3% interest *compounded continuously*.

What will be the balance after 7 years. (Exact answer in symbols, then decimal approximation.)

Solution $P=1000$ $r=.03$ $t=7$ A is unknown find A.

$$A = \underbrace{1000e^{(.03\cdot 7)}}_{\text{exact answer}} \approx \underbrace{\$1233^{\underline{68}}}_{\text{decimal approximation}}$$

Solving for Different Variables

The equation $A = Pe^{(rt)}$ involves four letters, A, P, r, t and is solved for A.

The equation can be solved for each of the letters P, r, t, as the next three examples show.

[Example 4] (Similar to 3.1#27B)

$937 is deposited into an account with 2.3% interest *compounded continuously*.

How long after initial deposit until the balance has grown to $1200?

(Give an exact answer in symbols, then a decimal approximation rounded to two decimal places.)

Solution $P = 937$ $r = .023$ $A = 1200$ t is unknown find t.

Solve $A = Pe^{(rt)}$ for t

Divide both sides by P

$$\frac{A}{P} = e^{(rt)}$$

Take natural logarithm of both sides

$$\ln\left(\frac{A}{P}\right) = \ln\left(e^{(rt)}\right) = rt$$

$$\ln\left(e^{(x)}\right) = x$$

divide both sides by r

$$t = \frac{\ln\left(\frac{A}{P}\right)}{r}$$

$$t = \frac{\ln\left(\frac{1200}{937}\right)}{0.023}$$ (exact answer) $\approx$ 10.75 years

use wolfram alpha

decimal approximation

[Example 5] (Similar to 3.1#35)

Deposit some money into account with 2.3% interest *compounded continuously*. How long until the balance doubles? (Exact answer in symbols, then decimal approximation.)

Solution

P is unknown

$r = .023$

A is unknown, but $A = 2P$ (balance doubles)

t is unknown find t

use form of the equation that is solved for t

$$t = \frac{\ln\left(\frac{A}{P}\right)}{r} = \frac{\ln\left(\frac{2P}{P}\right)}{0.023} = \frac{\ln(2)}{.023} \approx 30.14 \text{ years}$$

can cancel $\frac{P}{P}$ because $P \neq 0$

exact answer

decimal approximation

wolfram alpha

[Example 6] (Similar to 3.1#37)

If you want an account with *continuously compounded interest* to double in 20 years, what interest rate will you need?

(The book would ask *at what nominal rate compounded continuously must the money be invested?*)

(Exact answer in symbols, then decimal approximation.)

Solution

P is unknown

r is unknown find r

A is unknown but A = 2P because the money doubles

t = 20 years

Solve the original equation $A = Pe^{(rt)}$ for r

$$\frac{A}{P} = e^{(rt)}$$

$$\ln\left(\frac{A}{P}\right) = \ln\left(e^{(rt)}\right) = rt$$

$$r = \frac{\ln\left(\frac{A}{P}\right)}{t}$$

$$r = \frac{\ln\left(\frac{2\cancel{P}}{\cancel{P}}\right)}{20} = \frac{\ln(2)}{20} \approx .0347$$

exact answer

used wolfram alpha

decimal approximation

interest rate 3.47%

End of Video