**Subject for this video: Radioactive Decay**

**Reading:**

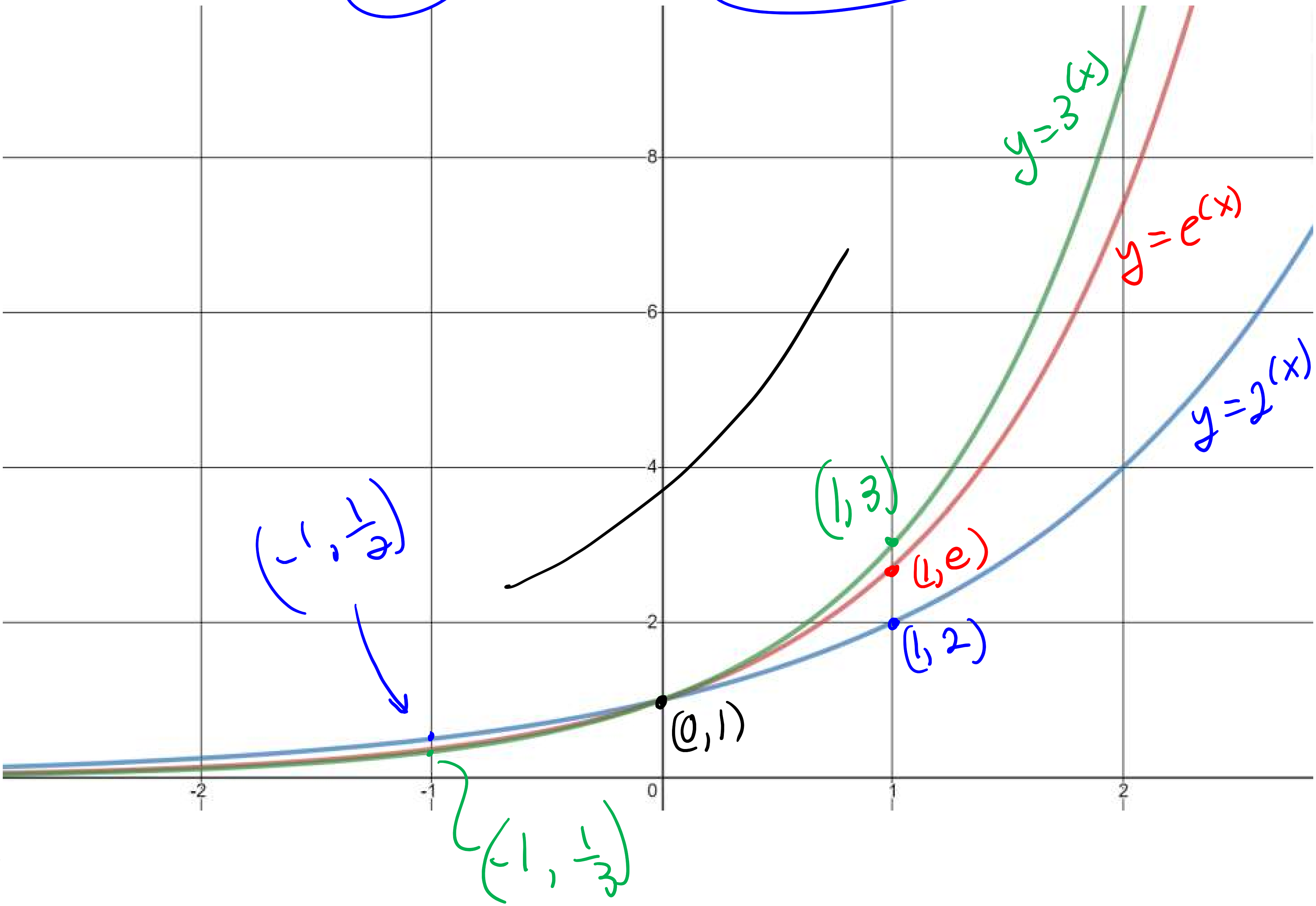
- **General:** Section 3.1 The Constant $e$ and Continuous Compound Interest
- **More Specifically:** This topic is only discussed in the introduction to Exercise #43

**Homework:**

**H38: Radioactive Decay (3.1#43,45)**

**Decaying Exponential Functions**

In the video for Homework H37, we drew the graphs of $2^{(x)}, 3^{(x)}, e^{(x)}$

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We also made a general observation about properties of one group of *exponential functions*.

**Properties of Exponential Functions $b^{(x)}$ with $b > 1$**

- Domain and Range
  - The domain is the set of all real numbers $x$. In interval notation, $(-\infty, \infty)$
  - The range is all $y > 0$. In interval notation, $(0, \infty)$
- The graph has three distinctive points:
  - The graph goes through the point $(x, y) = (0,1)$ because $b^{(0)} = 1$
  - The graph goes through the point $(x, y) = (1, b)$ because $b^{(b)} = b$
  - The graph goes through the point $(x, y) = \left(-1, \frac{1}{b}\right)$ because $b^{(-1)} = \frac{1}{b}$
- End Behavior
  - The graph goes up without bound on the right. That is, $\lim\limits_{x \to \infty} b^{(x)} = \infty$
  - Graph has a horizontal asymptote on left with equation $y = 0$. That is, $\lim\limits_{x \to -\infty} b^{(x)} = 0$
- The graph is increasing from left to right. That is, if $x_1 < x_2$, then $b^{(x_1)} < b^{(x_2)}$

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There is another group of exponential functions: the functions $b^{(x)}$ with base $b$ where $0 < b < 1$.

As a typical example, let's build a table of values for $y = \left(\frac{1}{2}\right)^{(x)}$ and examine its graph.

For comparison, we also build the table of values for $y = 2^{(x)}$ and show its graph.

| $\boldsymbol{x}$ | $\mathbf{2^{(x)}}$ | $\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{(x)}$ |
|---|---|---|
| **−3** | $2^{(-3)} = \frac{1}{2^3} = \frac{1}{8}$ | $\left(\frac{1}{2}\right)^{(-3)} = \frac{1}{(1/2)^3} = \frac{1}{(1/8)} = 8$ |
| **−2** | $2^{(-2)} = \frac{1}{2^{(2)}} = \frac{1}{4}$ | $\left(\frac{1}{2}\right)^{-2} = \frac{1}{(1/2)^2} = \frac{1}{(1/4)} = 4$ |
| **−1** | $2^{(-1)} = \frac{1}{2^1} = \frac{1}{2}$ | $\left(\frac{1}{2}\right)^{(-1)} = \frac{1}{(1/2)^1} = 2$ |
| **0** | $2^{(0)} = 1$ | $\left(\frac{1}{2}\right)^0 = 1$ |
| **1** | $2^{(1)} = 2$ | $\left(\frac{1}{2}\right)^1 = \frac{1}{2}$ |
| **2** | $2^{(2)} = 4$ | $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ |
| **3** | $2^{(3)} = 8$ | $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ |

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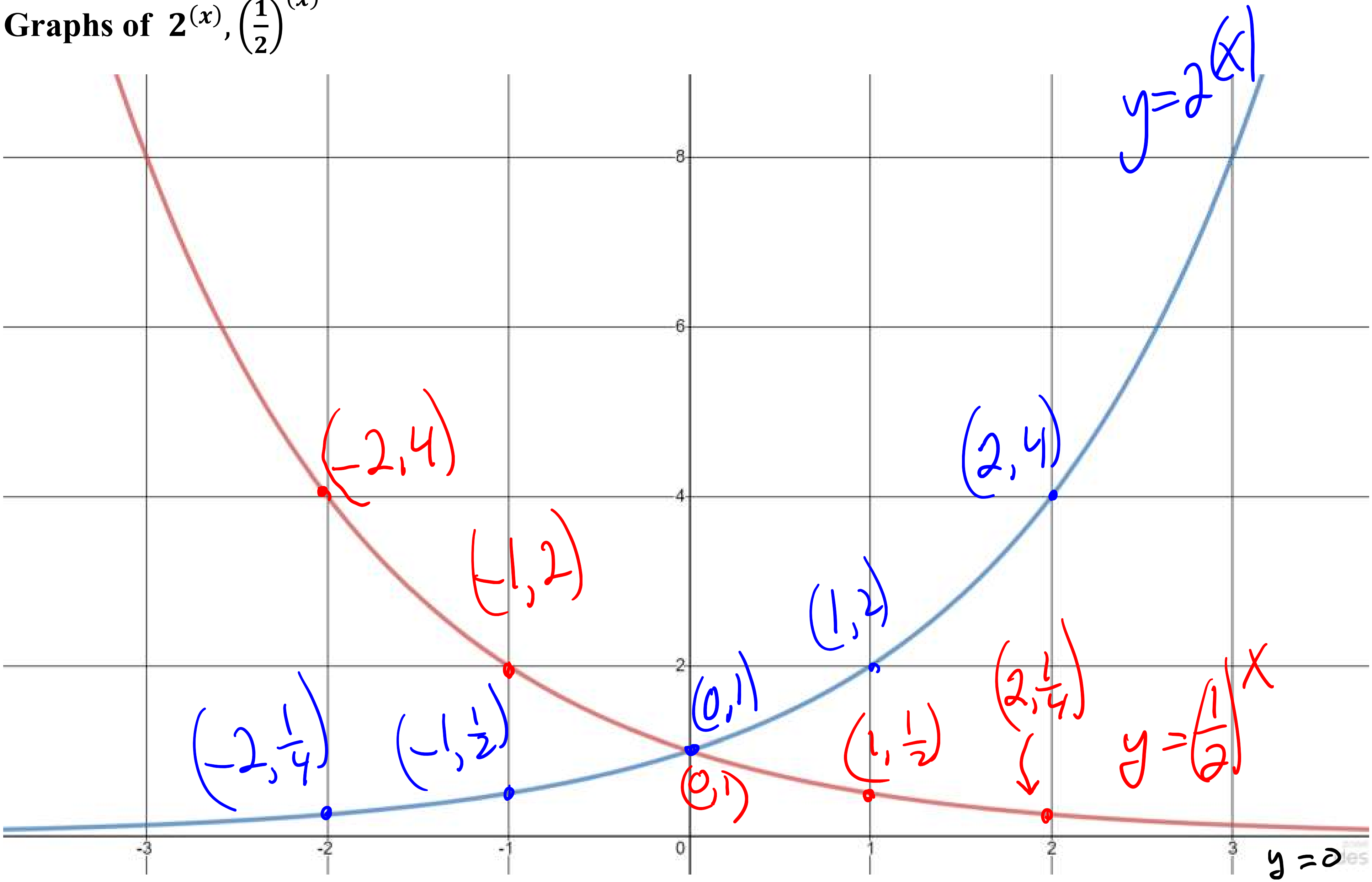
**Graphs of $2^{(x)}, \left(\frac{1}{2}\right)^{(x)}$**



We can make a general observation about properties of another group of *exponential functions*.

**Properties of Exponential Functions $b^{(x)}$ with $0 < b < 1$ ("*decaying*" exponentials)**

- Domain and Range
  - The domain is the set of all real numbers $x$. In interval notation, $(-\infty, \infty)$
  - The range is all $y > 0$. In interval notation, $(0, \infty)$
- The graph has three distinctive points:
  - The graph goes through the point $(x, y) = (0,1)$ because $b^{(0)} = 1$
  - The graph goes through the point $(x, y) = (1, b)$ because $b^{(1)} = b$
  - The graph goes through the point $(x, y) = \left(-1, \frac{1}{b}\right)$ because $b^{(-1)} = \frac{1}{b}$
- End Behavior
  - Graph has a horizontal asymptote on right with equation $y = 0$. That is, $\lim_{x\to\infty} b^{(x)} = 0$
  - The graph goes up without bound on the left. That is, $\lim_{x\to-\infty} b^{(x)} = \infty$
- The graph is decreasing from left to right. That is, if $x_1 < x_2$, then $b^{(x_1)} > b^{(x_2)}$

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Observe that the equation $y = \left(\frac{1}{2}\right)^{(x)}$ can be rewritten.

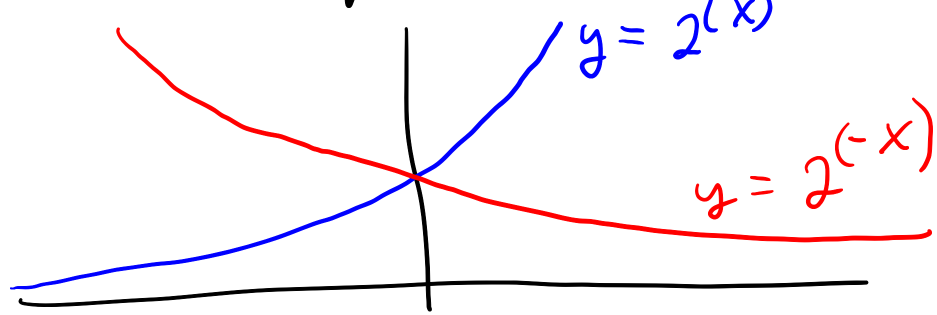
$$y = \left(\frac{1}{2}\right)^{(x)} = \left(2^{(-1)}\right)^{(x)} = 2^{(-x)}$$

because $\frac{1}{a} = a^{-1}$

because $(a^b)^c = a^{b \cdot c}$

Rewritten this way, we see that the graph makes sense from the perspective of transformations of graphs. That is, given a graph of $y = 2^{(x)}$, one can obtain a graph of $y = 2^{(-x)}$ by

flipping the blue graph across the y axis



More generally, if the base $b$ is a number $b > 1$, then the function $y = b^{(rx)}$

- will be an *increasing exponential function* when $r$ is a positive number,
- will be a *decaying exponential function* when $r$ is a negative number.

One place where decaying exponential functions show up is in models for *radioactive decay*.

**Radioactive Decay**

The decay of radioactive substances is described by the equation

$$Q = Q_0 e^{(rt)} \quad with \quad t \geq 0$$

In this equation,

$Q_0$ is the amount of the substance at time $t = 0$.

$r$ is the *continuous compound rate of decay*. (The number $r$ will be *negative*.)

$t$ is the *time* in years

$Q$ is the amount of the substance at time $t$.

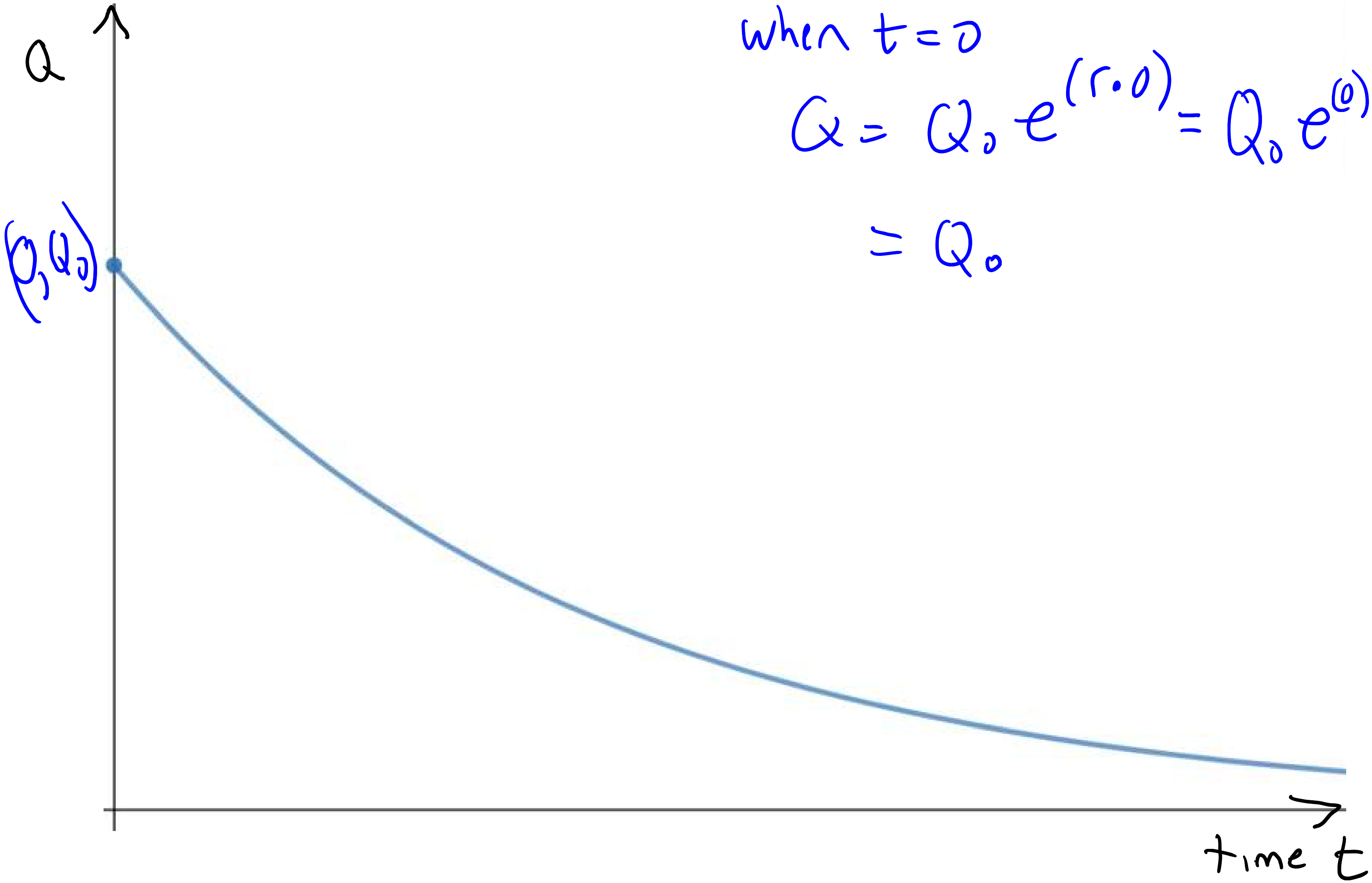
The graph of $Q - vs - t$ for radioactive decay

$$Q = Q_0 e^{(rt)}$$

when $t = 0$

$$Q = Q_0 e^{(r \cdot 0)} = Q_0 e^{(0)} = Q_0 \cdot 1$$

$$= Q_0$$



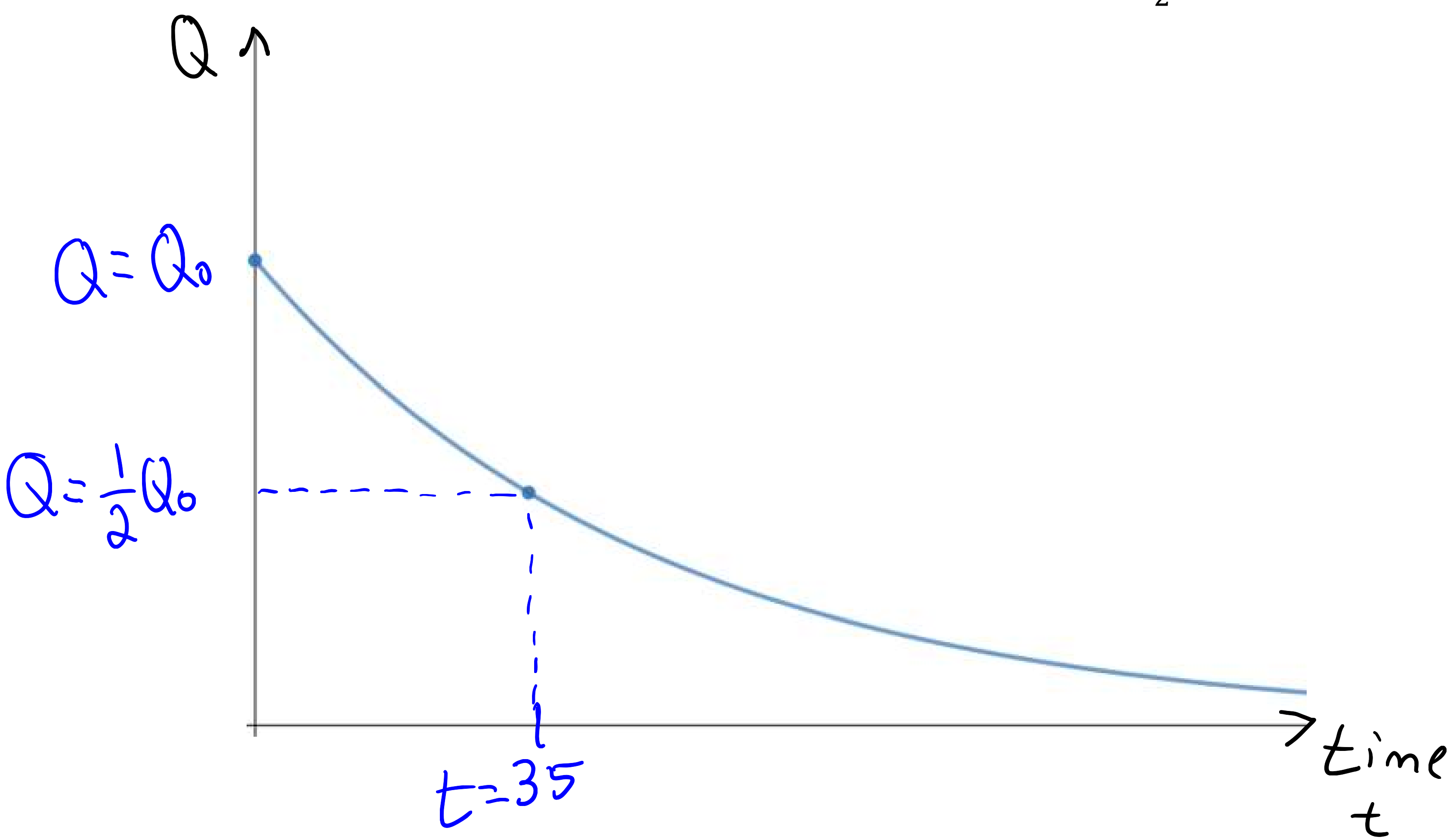

$t \geq 0$

**Half Life**

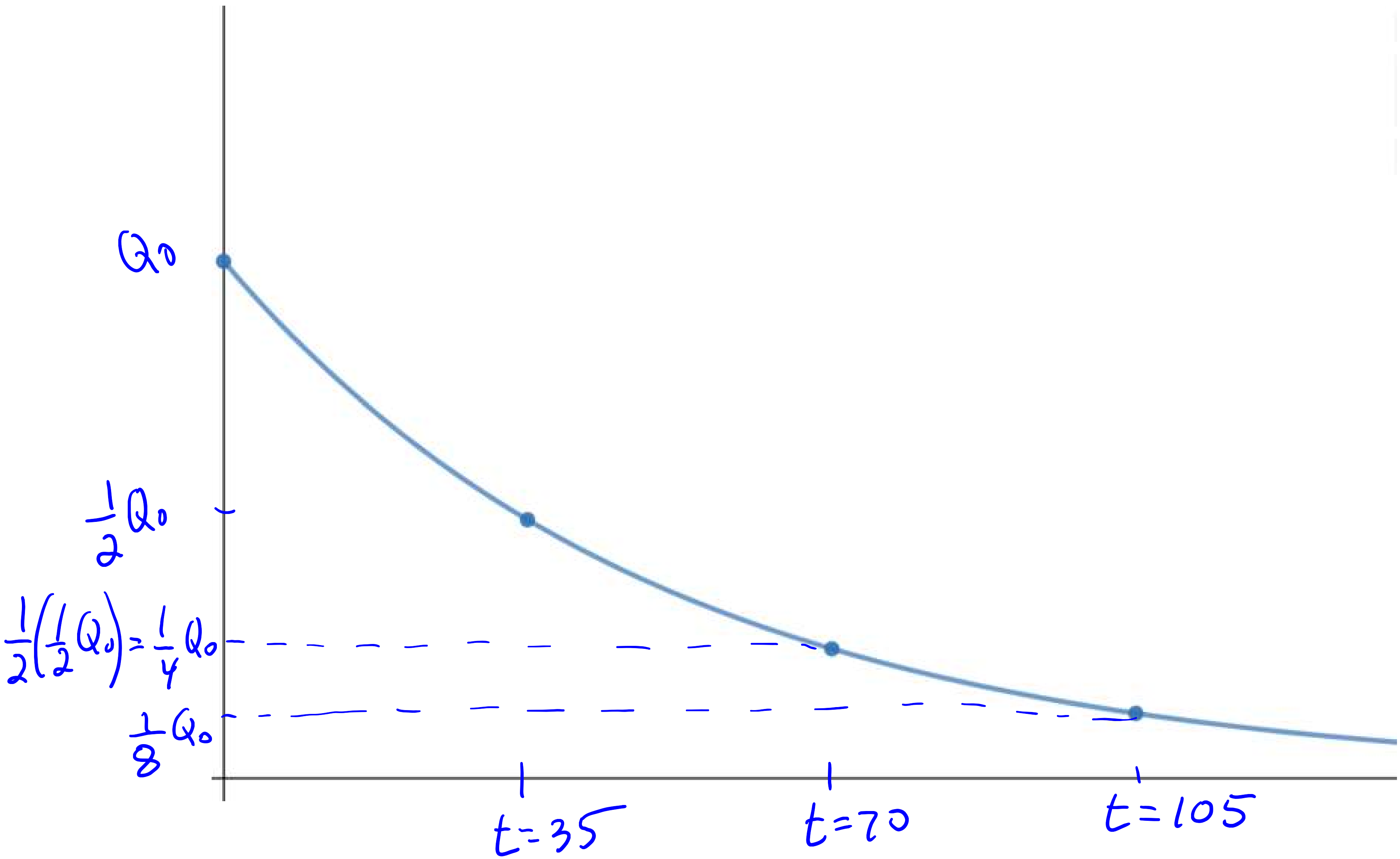
The *half-life* of a radioactive substance is the length of time required for the substance to decay to half its original amount.

For instance, suppose the amount of a substance at time $t = 0$ is $Q_0$ and its half life is 35 years.

Then the amount of the substance remaining at time $t = 35$ will be $Q = \frac{1}{2} Q_0$



Note that after another 35 years, at time $t = 70$, the substance will have decayed to half of the amount that was present at time $t = 35$, and so on.



**[Example 1]** (similar to 3.1#43)

The *continuous compound rate of decay* of the radioactive substance *carbon-14* is the number

$$r = -0.0001238$$

How long will it take a certain amount of carbon-14 to decay to half the original amount? That is, what is the *half-life* of *carbon-14*?

(Give an exact answer in symbols, then a decimal approximation rounded up to the nearest year.)

$$Q = Q_0 e^{(rt)}$$

$r = -0.0001238$

$Q_0$ is unknown

$Q$ is unknown but we know that $Q = \left(\frac{1}{2}\right) Q_0$

$t$ is unknown find $t$

Solve the equation for $t$

$$Q = Q_0 e^{(rt)}$$

Divide by $Q_0$

$$\frac{Q}{Q_0} = e^{(rt)}$$

take natural logarithm of both sides

$$\ln\left(\frac{Q}{Q_0}\right) = \ln\left(e^{(rt)}\right) = r \cdot t$$

because $\ln(e^{(x)}) = x$

divide both sides by $r$

$$t = \frac{\ln\left(\frac{Q}{Q_0}\right)}{r}$$

Substitute in the values that we know

$$t = \frac{\ln\left(\frac{\left(\frac{1}{2}\right)\cancel{Q_0}}{\cancel{Q_0}}\right)}{-.0001238} = \frac{\ln\left(\frac{1}{2}\right)}{-.0001238} \approx 5599 \text{ years}$$

exact answer

use wolfram alpha

decimal approximation

**[Example 2]** (similar to 3.1#45)

A strontium isotope has a half-life of 90 years. What is the *continuous compound rate of decay*?

(Give an exact answer in symbols, then a decimal approximation rounded to 4 significant digits.)

Solution

$Q_0$ is unknown

$r$ is unknown (find $r$)

$Q$ is unknown, but we know $Q = \frac{1}{2} Q_0$

$t = 90$

Solve the equation for $r$

**End of [Example 2]**

**End of Video**

$$Q = Q_0 e^{(rt)}$$

divide by $Q_0$

$$\frac{Q}{Q_0} = e^{(rt)}$$

take natural logarithm of both sides

$$\ln\left(\frac{Q}{Q_0}\right) = \ln\left(e^{(rt)}\right) = rt$$

Divide by $t$

$$r = \frac{\ln\left(\frac{Q}{Q_0}\right)}{t}$$

Substitute in our known values

$$r = \frac{\ln\left(\frac{\left(\frac{1}{2}\right)\cancel{Q_0}}{\cancel{Q_0}}\right)}{90} = \underbrace{\frac{\ln\left(\frac{1}{2}\right)}{90}}_{\text{exact answer}} \approx \underbrace{-0.007202}_{\text{decimal approximation}}$$

used wolfram alpha.

four significant digits