Subject for this video: Derivatives of Exponential Functions

Reading:

- **General:** Section 3.2 Derivatives of Exponential and Logarithmic Functinos
- **More Specifically:** page 187 – 188 and Example 1, page 191 – 192 and Example 3A

Homework:

H40: Differentiating Exponential Functions (3.2#13,28,49, 57)

Recall the Derivative Rules that we learned about in the previous videos.

The Constant Function Rule

This rule is used for finding the derivative of a *constant* function.

Two equation form: If $f(x) = c$ then $f'(x) = 0$.

Single equation form: $\frac{d}{dx} c = 0$

The Power Rule

This rule is used for finding the derivative of a *power* function.

Two equation form: If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Single equation form: $\frac{d}{dx} x^n = nx^{n-1}$

The Sum and Constant Multiple Rule

If $f(x)$ and $g(x)$ are functions and a, b are constants, then

$$\frac{d}{dx}\big(af(x) + bg(x)\big) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$

Using prime notation, we could write

$$\big(af(x) + bg(x)\big)' = af'(x) + bg'(x)$$

In this video, we will add three more Rules, for finding derivatives of exponential functions. The first rule is about the derivative of $y = e^{(x)}$.

Exponential Function Rule #1

This rule is used for finding the derivative of the *base e exponential function*.

Two equation form: If $f(x) = e^{(x)}$ then $f'(x) = e^{(x)}$.

Single equation form: $\frac{d}{dx} e^{(x)} = e^{(x)}$

This wonderfully simple rule is found by using the *Definition of the Derivative*.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{(x+h)} - e^{(x)}}{h}$$

An early key step in the computation uses an old fact about exponents, that $e^{(x+h)} = e^{(x)} e^{(h)}$

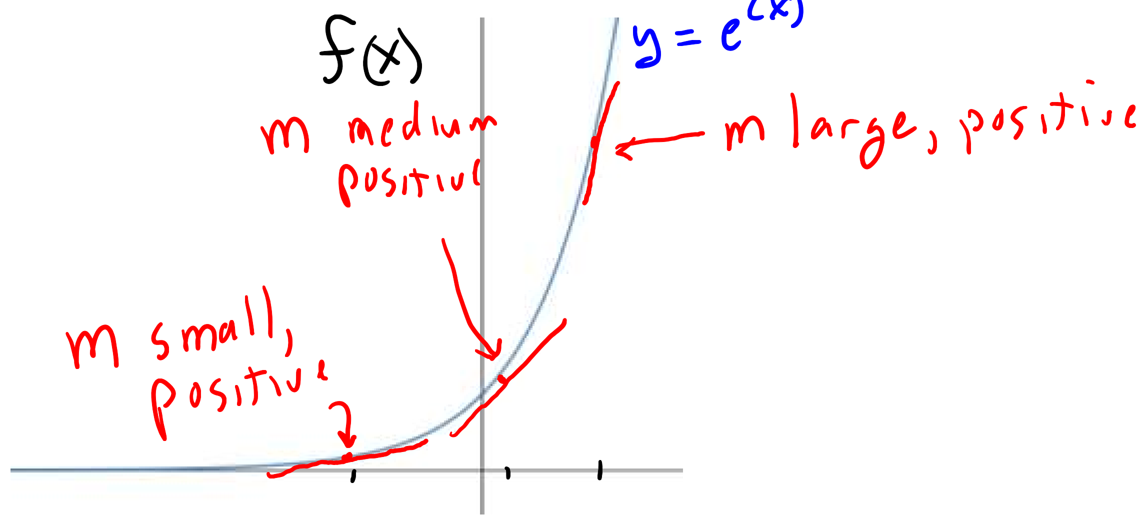
A later key step uses a fact from higher math

$$\lim_{h \to 0} \frac{e^{(h)} - 1}{h} = 1$$

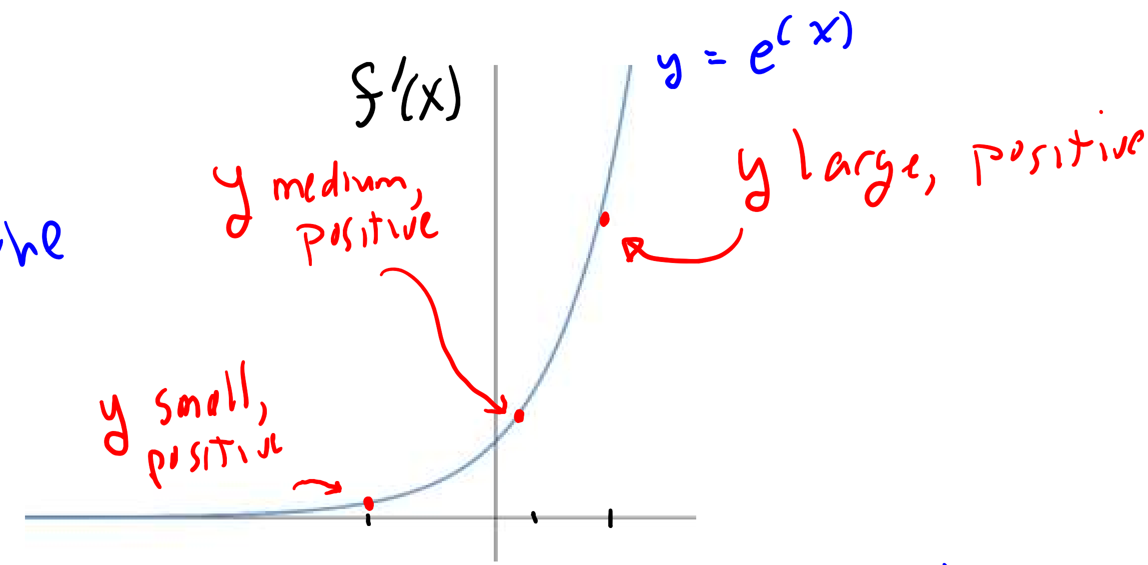
The details of the calculation, organized by the *Four Step Process*, are shown clearly in the book on pages 187 – 188. It is interesting, beautiful math, worth reading. But in MATH 1350, you will not be asked to do the calculation, and I won't discuss the details here.

But we will gain a better *understanding* of the rule if we examine the *graphs* of $f(x)$ and $f'(x)$.

In the video for Homork H37, we saw that it is possible to draw the graph of $y = e^{(x)}$ by hand, using the graphs of $y = 2^{(x)}$ and $y = 3^{(x)}$ as guides. Here we will just use a computer graph.



Observe that the numbers that are the slopes on the graph of $f(x)$



agree with the numbers that are the y values on the graph of $f'(x)$

So it is believable that $\frac{d}{dx} e^{(x)} = e^{(x)}$

Exponential Function Rule #2 is very similar to Exponential Function Rule #1.

Exponential Function Rule #2

Two equation form: If $f(x) = e^{(kx)}$ then $f'(x) = ke^{(kx)}$.

Single equation form: $\frac{d}{dx}e^{(kx)} = ke^{(kx)}$

Rule #2 is also found by using the *Definition of the Derivative*, using a slight variation on the computation used to find Rule #1.

The book discusses this new derivative result only in exercises 3.2 # 61, 62. The book does not discuss the result in the reading and never presents it in a list of derivative rules. That is a shame, because it is one of the most-used derivative rules. That's why I have given it the name Rule #2 and put it in a green box. We will be using it a lot in the future.

[Example 1] Use Exponential Function Rule #2 to find the derivative of $y = b^x$.

Solution

Start by rewriting the function

$$y = b^{(x)} = \left(e^{(\ln(b))}\right)^{(x)} = e^{(\ln(b)\cdot x)}$$

because $b = e^{(\ln(b))}$ rule of exponents

because $(e^m)^n = e^{m \cdot n}$ rule of exponents

$$\text{So } \frac{dy}{dx} = \frac{d}{dx} e^{(\ln(b)\cdot x)} = \ln(b)\cdot e^{(\ln(b)\cdot x)} = \ln(b)\cdot b^{(x)}$$

use rule #2 with $k = \ln(b)$

using fact that $e^{(\ln(b)\cdot x)} = b^{(x)}$

$$\boxed{\frac{d}{dx} b^{(x)} = \ln(b)\cdot b^{(x)}}$$

Our result from **[Example 1]** amounts to a new Derivative Rule!

Exponential Function Rule #3

Two equation form: If $f(x) = b^{(x)}$ then $f'(x) = b^{(x)} \cdot \ln(b)$.

Single equation form: $\frac{d}{dx} b^{(x)} = b^{(x)} \cdot \ln(b)$

[Example 2] Derivatives of Basic Functions Involving Exponents

For each function, find $f'(x)$

(A) $f(x) = 5e^{(x)}$

$$f'(x) = \frac{d}{dx} 5e^{(x)} = 5 \frac{d}{dx} e^{(x)} = 5 \cdot e^{(x)}$$

use constant multiple rule

use exponential function rule #1

(B) $f(x) = 5e^{(7x)}$

$$f'(x) = \frac{d}{dx} 5e^{(7x)} = 5 \frac{d}{dx} e^{(7x)} = 5 \cdot 7e^{(7x)} = 35e^{(7x)}$$

constant multiple rule

use rule #2 with $k = 7$

simplify

(C) $f(x) = 5 \cdot 7^{(x)}$

$$f'(x) = \frac{d}{dx} 5 \cdot 7^{(x)} = 5 \cdot \frac{d}{dx} 7^{(x)} = 5 \cdot \left(7^{(x)} \cdot \ln(7) \right) = 5 \cdot \ln(7) \cdot 7^{(x)}$$

constant multiple rule

rule #3 with b=7

clean up

(D) $f(x) = 5e^{(7)}$

$$f'(x) = \frac{d}{dx} \underline{5e^{(7)}} = 0$$

constant

constant function rule

(E) $f(x) = 5x^{e}$

power function with n=e

$$f'(x) = \frac{d}{dx} 5 x^{e} = 5 \frac{d}{dx} x^{e} = 5 \left(e\, x^{e-1} \right) = \underline{5e} \cdot x^{e-1}$$

constant multiple rule

use power rule with n=e

clean up

[Example 3] Derivatives of More Complicated Functions Involving Exponents

(A) Differentiate $7 - 5x + 13e^{(x)}$

$$\frac{d}{dx}\left(7 - 5x + 13e^{(x)}\right) = \frac{d}{dx}7 - 5\frac{d}{dx}x + 13\frac{d}{dx}e^{(x)} = 0 - 5(1\cdot x^{1-1}) + 13(e^{(x)})$$

$$= -5 + 13e^{(x)}$$

constant function rule

rule#1

sum and constant multiple rule

power rule with n=1

(B) Find f' for $f(x) = 5x^e - 7e^{(3x)}$

$$f'(x) = \frac{d}{dx}\left(5x^e - 7e^{(3x)}\right) = 5\frac{d}{dx}e^{(x)} - 7\frac{d}{dx}e^{(3x)} = 5(e^{(x)}) - 7(3\cdot e^{(3x)})$$

$$= 5e^{(x)} - 21e^{(3x)}$$

mistake here

rule#1

sum and constant multiple rule

rule#2 with k=3

See next page for corrected calculation

(C) Find $\frac{dy}{dx}$ for the function $y = 3^{(x)} + x^3 + e^3$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3^{(x)} + x^3 + e^3\right) = \frac{d}{dx}3^{(x)} + \frac{d}{dx}x^3 + \frac{d}{dx}e^3 = 3^{(x)}\cdot\ln(3) + 3\cdot x^{3-1} + 0$$

rule#3 with b=3

sum rule

power rule with n=3

constant function rule

$$= 3^{(x)}\cdot\ln(3) + 3x^2$$

Corrected Calculation for [Example 3] (B)

Rule #2

$$f'(x) = \frac{d}{dx}\left(5x^e - 7e^{(3x)}\right) = 5\frac{d}{dx}x^e - 7\frac{d}{dx}e^{(3x)} = 5\left(e \cdot x^{e-1}\right) - 7\left(3 \cdot e^{(3x)}\right)$$

Sum and Constant Multiple Rule

Power Rule with $n=e$

$$= 5 \cdot e \cdot x^{e-1} - 21 e^{(3x)}$$

The yellow shaded term was incorrect on the previous page