Subject for this video: Derivatives of Logarithmic Functions

Reading:

- **General:** Section 3.2 Derivatives of Exponential and Logarithmic Functions
- **More Specifically:** Bottom of p. 188 – middle of p. 192, Examples 2 & 3.

Homework:

H43: Derivatives of Logarithmic Functions (3.2#15,21,43,44,51,55)

Prerequisite Skills: Recall these Properties of Logarithms:

Logarithm of a Product:	$\ln(a \cdot b) = \ln(a) + \ln(b)$
Logarithm of an Exponential Expression:	$\ln(a^b) = b\ln(a)$
Logarithm of a Reciprocal:	$\ln\left(\frac{1}{a}\right) = -\ln(a)$
Logarithm of a Quotient:	$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

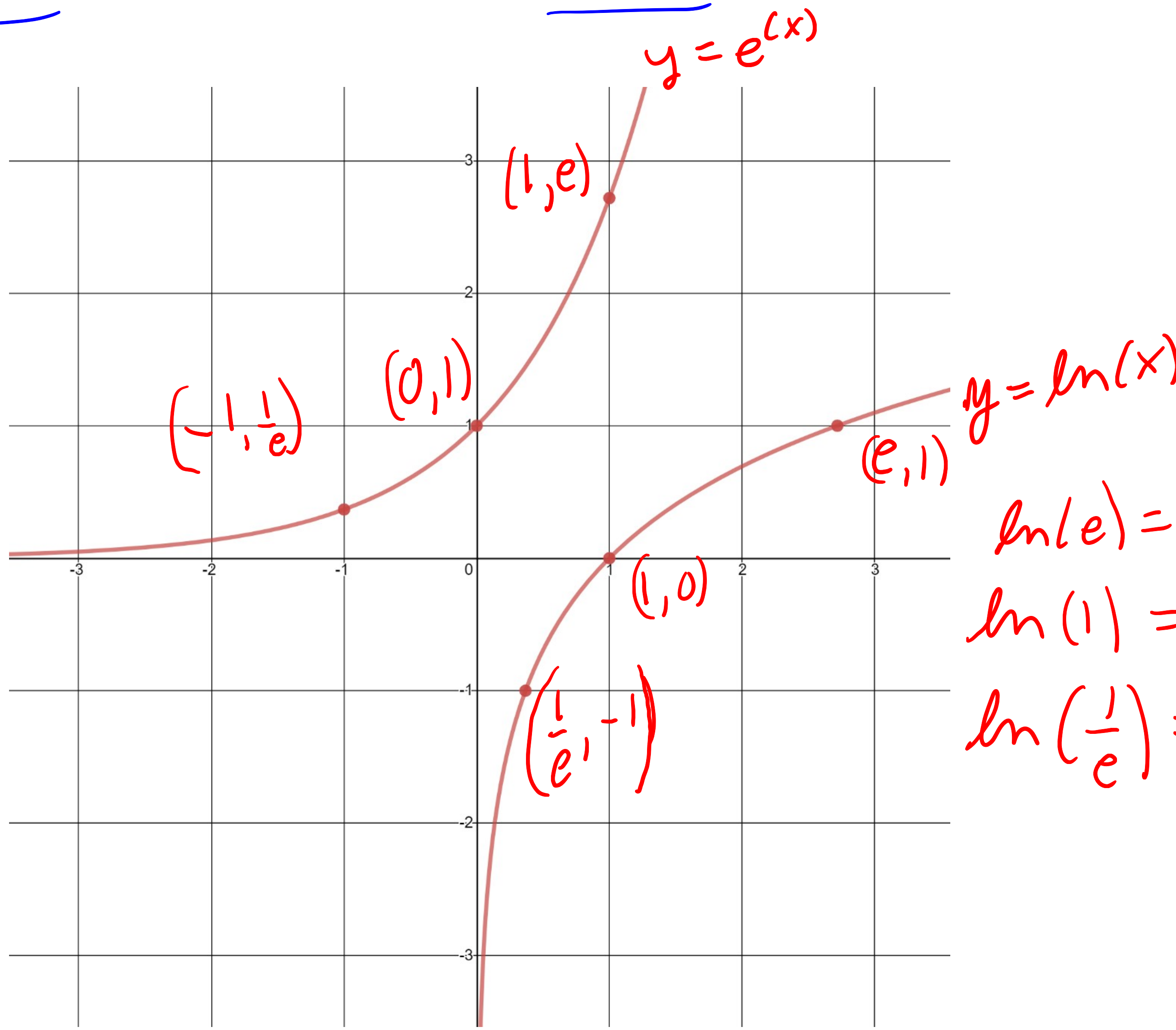
Prerequisite Skills: Recall the shape of the graph of $y = \ln(x)$.

The graph of $y = \ln(x)$ is obtained from the graph of $y = e^{(x)}$ by interchanging all the x, y values.

$e^{(1)} = e$

$e^{(0)} = 1$

$e^{(-1)} = \frac{1}{e}$



$\ln(e) = 1$

$\ln(1) = 0$

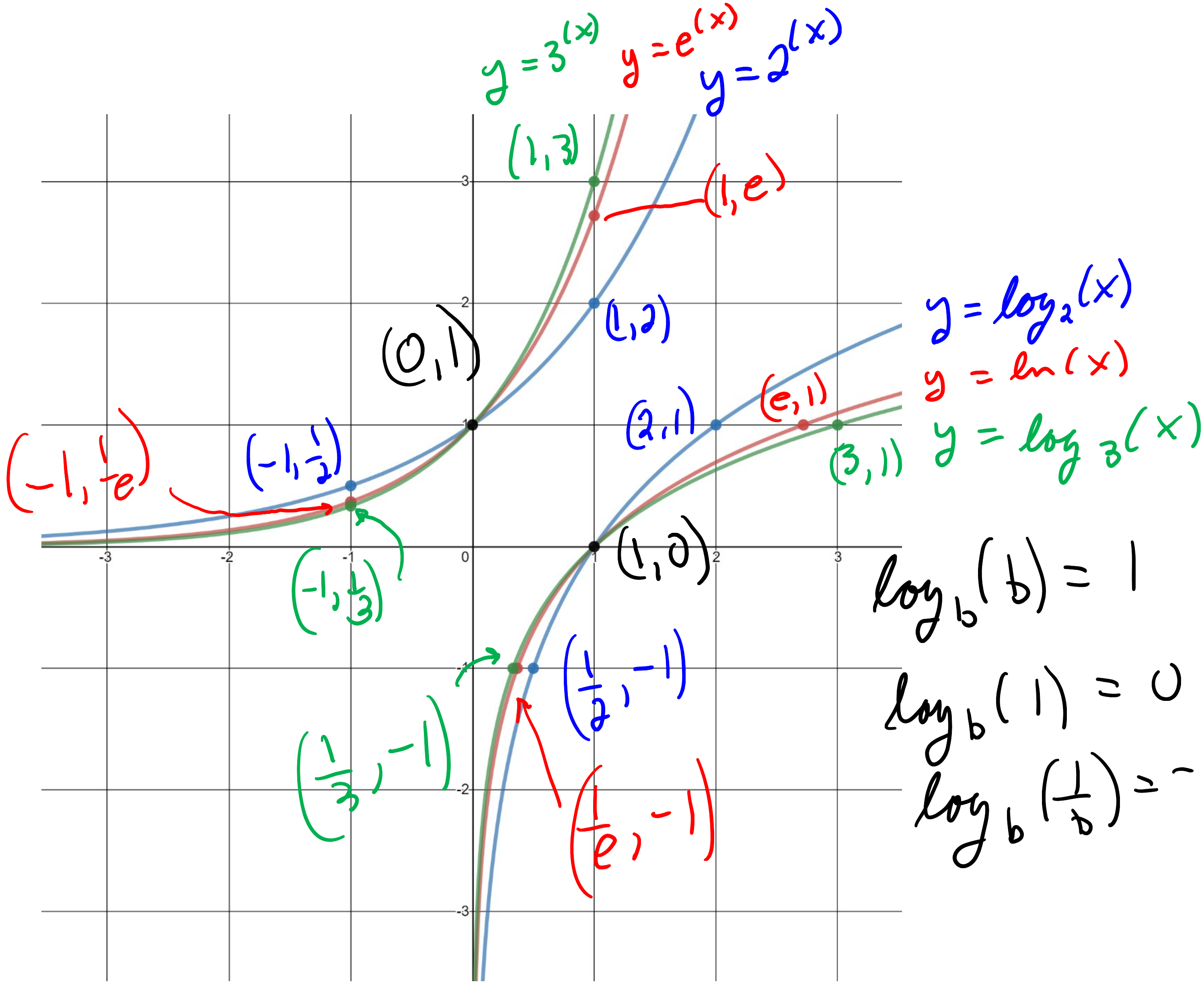
$\ln\left(\frac{1}{e}\right) = -1$

More generally, the graph of $y = \log_b(x)$ is obtained from the graph of $y = b^{(x)}$ by interchanging all the x, y values.

$b^{(1)} = b$

$b^{(0)} = 1$

$b^{(-1)} = \frac{1}{b}$



$\log_b(b) = 1$

$\log_b(1) = 0$

$\log_b\left(\frac{1}{b}\right) = -1$

Recall from the video for H32 these observations about properties of *exponential functions*.

Properties of Exponential Functions $b^{(x)}$ with $b > 1$

- Domain and Range
  - The domain is the set of all real numbers x. In interval notation, $(-\infty, \infty)$
  - The range is all $y > 0$. In interval notation, $(0, \infty)$
- The graph has three distinctive points:
  - The graph goes through the point $(x, y) = (0,1)$ because $b^{(0)} = 1$
  - The graph goes through the point $(x, y) = (1, b)$ because $b^{(b)} = b$
  - The graph goes through the point $(x, y) = \left(-1, \frac{1}{b}\right)$ because $b^{(-1)} = \frac{1}{b}$
- End Behavior
  - The graph goes up without bound on the right. That is, $\lim_{x\to\infty} b^{(x)} = \infty$
  - Graph has a horizontal asymptote on left with equation $y = 0$. That is, $\lim_{x\to-\infty} b^{(x)} = 0$
- The graph is increasing from left to right. That is, if $x_1 < x_2$, then $b^{(x_1)} < b^{(x_2)}$

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We can make corresponding observations about properties of *exponential functions*.

Properties of Logarithmic Functions $\log_b(x)$ with $b > 1$

- Domain and Range
  - The domain is the set of all positive real numbers x. In interval notation, $(0, \infty)$
  - The range is all real numbers y. In interval notation, $(-\infty, \infty)$
- The graph has three distinctive points:
  - The graph goes through the point $(x, y) = (1,0)$, which tells us $\log_b(1) = 0$
  - The graph goes through the point $(x, y) = (b, 1)$, which tells us $\log_b(b) = 1$
  - The graph goes through the point $(x, y) = \left(\frac{1}{b}, -1\right)$, which tells us $\log_b\left(\frac{1}{b}\right) = -1$
- End Behavior
  - The graph goes up without bound on the right. That is, $\lim_{x\to\infty} \log_b(x) = \infty$
  - Graph has a vertical asymptote with line equation $x = 0$, and the graph goes down along the right side of that asymptote. That is, $\lim_{x\to 0^+} \log_b(x) = -\infty$
- The graph is increasing from left to right. That is, if $x_1 < x_2$, then $\log_b(x_1) < \log_b(x_2)$

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New Derivative Rules

Logarithmic Function Rule #1

This rule is used for finding the derivative of the *natural logarithm function*.

Two equation form: If $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$.

Single equation form: $\frac{d}{dx}\ln(x) = \frac{1}{x}$

note $\ln(x) \neq \frac{1}{x}$!!

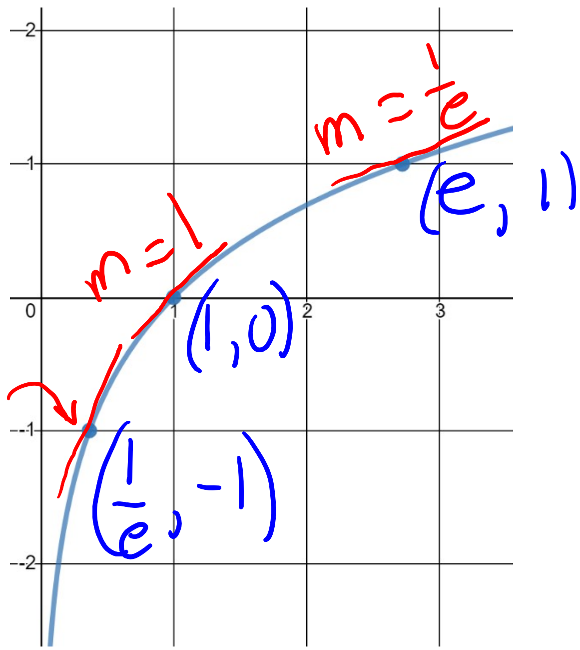
This rule is found by using the *Definition of the Derivative*.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}$$

The details of the calculation, organized by the *Four Step Process*, are shown clearly in the book on pages 189 – 190. It is interesting, beautiful math, worth reading. But in MATH 1350, you will not be asked to do the calculation, and I won't discuss the details here.

But we will gain a better *understanding* of the rule if we examine the *graphs* of $f(x)$ and $f'(x)$.

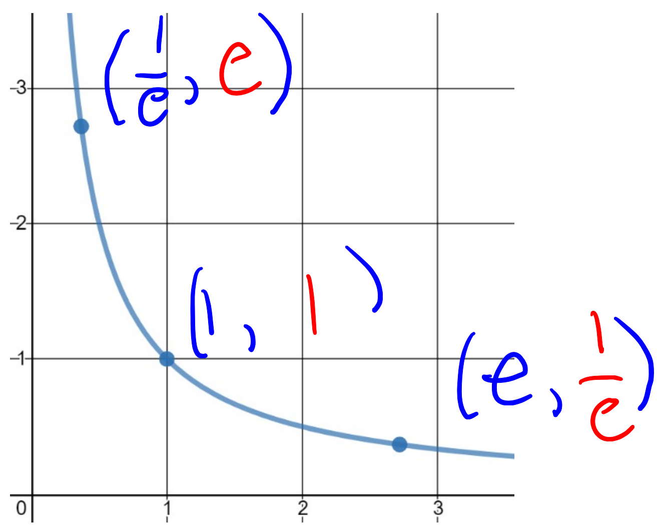
$f(x) = \ln(x)$



The numbers that are the slopes on the graph of $f(x) = \ln(x)$

are equal to

$f'(x) = \frac{1}{x}$



the numbers that are the y values on the graph of $f'(x) = \frac{1}{x}$

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There is a second Logarithmic Function Rule

Logarithmic Function Rule #2

This rule is used for finding the derivative of *base b logarithm functions*.

Two equation form: If $f(x) = \log_b(x)$ then $f'(x) = \frac{1}{x\ln(b)}$.

Single equation form: $\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$

Observe that Logarithmic Function Rule #1 is a special case of Logarthmic Function Rule #2

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\log_e(x) = \frac{1}{x\ln(e)} = \frac{1}{x}$$

because $\ln(x)$ means $\log_e(x)$

Logarithm Function Rule #2

because $\ln(e) = 1$

[Examples] Find the derivatives of the following functions:

(A) $f(x) = 12\ln(x)$

$$f'(x) = \frac{d}{dx}\underline{12}\ln(x) = \underline{12}\frac{d}{dx}\ln(x) = 12\left(\frac{1}{x}\right) = \frac{12}{x}$$

Constant multiple rule

Logarithm Rule #1

clean up

(B) $f(x) = 12\log_{13}(x)$

$$f'(x) = \frac{d}{dx}\underline{12}\log_{13}(x) = \underline{12}\frac{d}{dx}\log_{13}(x) = 12\left(\frac{1}{x\ln(13)}\right) = \frac{12}{x\ln(13)}$$

constant multiple rule

logarithm rule #2

(C) $f(x) = 12\log(x)$

In our book, $\log(x)$ means $\log_{10}(x)$

(Warning: In some books and in some computer programs, $\log(x)$ means $\ln(x)$.)

$$f'(x) = \frac{d}{dx}12\log_{10}(x) = \ldots = \frac{12}{x\ln(10)}$$

Steps just like (B)

(D) $f(x) = 12\ln(13)$

$$f'(x) = \frac{d}{dx}\underbrace{12\ln(13)}_{\text{constant function}} = 0$$

(E) $f(x) = 12\ln(13x)$ Neither one of our rules will work on $\ln(13x)$

Must start by rewriting $f(x)$ into a form that works with our derivative rules.

$$f(x) = 12\ln(13x) = 12\left(\ln(13) + \ln(x)\right) = 12\ln(13) + 12\ln(x)$$

property $\ln(a \cdot b) = \ln(a) + \ln(b)$

$$f'(x) = \frac{d}{dx}12\ln(13) + \frac{d}{dx}12\ln(x) = 0 + \frac{12}{x} = \frac{12}{x}$$

from A

from (D)

(F) $f(x) = 12\ln\left(\frac{13}{x}\right)$

Start by rewriting

$$f(x) = 12\ln\left(\frac{13}{x}\right) = 12\left[\ln(13) - \ln(x)\right] = 12\ln(13) - 12\ln(x)$$

property $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$$f'(x) = \frac{d}{dx}12\ln(13) - \frac{d}{dx}12\ln(x) = 0 - \frac{12}{x} = -\frac{12}{x}$$

from A

from D

(G) $f(x) = 12\ln(x^{13})$

Rewrite $f(x) = 12\ln(x^{13}) = 12\cdot 13\ln(x)$

use property $\ln(a^b) = b\ln(a)$

$$f'(x) = \frac{d}{dx}\, 12\cdot 13\ln(x) = \underline{12\cdot 13}\cdot\frac{d}{dx}\ln(x) = 12\cdot 13\cdot\frac{1}{x} = \frac{156}{x}$$

constant multiple rule

rule #1

(H) $f(x) = 12x\ln(13)$

Power function with $n=1$

$$f'(x) = \frac{d}{dx}\, \underline{12}\, x\, \underline{\ln(13)} = \underline{12}\,\underline{\ln(13)}\frac{d}{dx}x = 12\ln(13)\cdot 1$$

constant multiple rule

power rule with $n=1$

$$= 12\ln(13)$$