Subject for this video: Derivatives of Quotients

Reading:

- **General:** Section 3.3 Derivatives of Products and Quotients
- **More Specifically:** Middle of pages 198 – top of page 201, Examples 4AB, 5

Homework:

H46: Derivatives of Quotients (3.3#25,31,33,69)

Recall the Derivative Rules that we learned about in previous videos.

The Constant Function Rule:	$\frac{d}{dx}c = 0$
The Power Rule:	$\frac{d}{dx}x^n = nx^{n-1}$
The Sum and Constant Multiple Rule:	$\frac{d}{dx}\big(af(x) + bg(x)\big) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$
Exponential Function Rule #1:	$\frac{d}{dx}e^{(x)} = e^{(x)}$
Exponential Function Rule #2:	$\frac{d}{dx}e^{(kx)} = ke^{(kx)}$
Exponential Function Rule #3:	$\frac{d}{dx}b^{(x)} = b^{(x)} \cdot \ln(b)$
Logarithmic Function Rule #1:	$\frac{d}{dx}\ln(x) = \frac{1}{x}$
Logarithmic Function Rule #2:	$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$
The Product Rule:	$\frac{d}{dx}g(x) \cdot h(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

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New Rule: The Quotient Rule

Consider a quotient of functions,

$$f(x) = \frac{top(x)}{bottom(x)}$$

Question: How is $f'(x)$ related to $top'(x)$ and $bottom'(x)$?

The good news is that there is an obvious relationship.

The bad news is that the obvious relationship is wrong.

The obvious relationship: $\cancel{\dfrac{\frac{d}{dx}\,\text{top}(x)}{\frac{d}{dx}\,\text{bottom}(x)} = \dfrac{\frac{d}{dx}\text{top}(x)}{\frac{d}{dx}\text{bottom}(x)}}$ Wrong!

How do we know it is wrong? Consider this easy example.

[***Example* 1**] *Let* $f(x) = \dfrac{x^2}{5}$

If we find $f'(x)$ using our established derivative rules, we find

Start by rewriting $f(x) = \dfrac{x^2}{5} = \left(\dfrac{1}{5}\right)\cdot x^2$ power function with $n=2$

$$f'(x) = \frac{d}{dx}\left(\left(\frac{1}{5}\right)x^2\right) \underset{\uparrow}{=} \left(\frac{1}{5}\right)\frac{d}{dx}x^2 \underset{\uparrow}{=} \left(\frac{1}{5}\right)(2x^1) = \frac{2x}{5}$$

constant multiple rule

power rule with $n=2$

But if we compute $f'(x)$ using the obvious (bad) method, we find

$$\cancel{f'(x) = \frac{d}{dx}\left(\frac{x^2}{5}\right) = \frac{\frac{d}{dx}x^2}{\frac{d}{dx}5} \overset{\text{power rule}}{=} \frac{2x}{0}}$$

Does not exist!!

wrong!

constant function rule

Wrong

The correct relationship between $f'(x)$ and $top'(x)$ and $bottom'(x)$ is given by the *Quotient Rule*.

The Quotient Rule

This rule is used for finding the derivative of a *quotient of functions*.

Two equation form:

If

$$f(x) = \frac{top(x)}{bottom(x)}$$

then

$$f'(x) = \frac{top'(x)bottom(x) - top(x)bottom'(x)}{\left(bottom(x)\right)^2}$$

Single equation form:

$$\frac{d}{dx}\left(\frac{top(x)}{bottom(x)}\right) = \frac{top'(x)bottom(x) - top(x)bottom'(x)}{\left(bottom(x)\right)^2}$$

[Example 2] Revisit our function from **[Example 1]** $f(x) = \frac{x^2}{5}$

Find $f'(x)$ using the ~~product~~ quotient rule. Simplify your answer.

$$f'(x) = \frac{top'(x) \cdot bottom(x) - top(x) \cdot bottom'(x)}{(bottom(x))^2}$$

$$\frac{d}{dx}\left(\frac{x^2}{5}\right) = \frac{\left(\frac{d}{dx}x^2\right) \cdot 5 - x^2\left(\frac{d}{dx}5\right)}{(5)^2}$$

quotient rule

power rule

constant function rule

$$= \frac{(2 \cdot x^1) \cdot 5 - \cancel{x^2(0)}}{25}$$

$\to 0$

$$= \frac{2x \cdot 5}{25}$$

simplify

$$= \frac{2x}{5}$$

Remark: Solution in [Example 1] using easier derivative rules is much simpler!
Always try to rewrite function first!

[Example 3] (similar to 3.3#25)

$$Let\ \ f(x) = \frac{3x+5}{x^2-3}$$

Find $f'(x)$ and simplify answer.

Solution: $$f'(x) = \frac{\left(\frac{d}{dx}3x+5\right)\cdot(x^2-3) - (3x+5)\left(\frac{d}{dx}(x^2-3)\right)}{(x^2-3)^2}$$

quotient rule

$$= \frac{(3(1)+0)\cdot(x^2-3) - (3x+5)(2x-0)}{(x^2-3)^2}$$

note: cannot cancel (x^2-3) at this point!

$$= \frac{3(x^2-3) - (3x+5)(2x)}{(x^2-3)^2}$$

$$= \frac{(3x^2-9) - (6x^2+10x)}{(x^2-3)^2}$$

$$= \frac{-3x^2-10x-9}{(x^2-3)^2}$$

[Example 4] (similar to 3.3#31)

Let $f(x) = \dfrac{e^{(x)}}{x^2 - 3}$

Find $f'(x)$ and simplify answer.

Solution: $f'(x) = \dfrac{d}{dx}\left(\dfrac{e^{(x)}}{x^2-3}\right) = \dfrac{\left(\frac{d}{dx} e^{(x)}\right)\cdot(x^2-3) - e^{(x)} \frac{d}{dx}(x^2-3)}{(x^2-3)^2}$

quotient rule

$= \dfrac{\left(e^{(x)}\right)\cdot(x^2-3) - e^{(x)}(2x)}{(x^2-3)^2}$

note we cannot cancel the (x^2-3) here!

Simplify numerator by factoring out the $e^{(x)}$

$= \dfrac{e^{(x)}\left((x^2-3) - (2x)\right)}{(x^2-3)^2}$

$= \dfrac{e^{(x)}(x^2 - 2x - 3)}{(x^2-3)^2}$

[Example 5] (similar to 3.3#33)

Let $f(x) = \dfrac{\ln(x)}{x^2 - 3x + 5}$

Find $f'(x)$ and simplify answer.

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

$$f'(x) = \frac{\left(\frac{d}{dx}\ln(x)\right)(x^2-3x+5) - \ln(x)\left(\frac{d}{dx}(x^2-3x+5)\right)}{(x^2-3x+5)^2}$$

quotient rule

note: cannot cancel the (x^2-3x+5)!

$$= \frac{\left(\frac{1}{x}\right)(x^2-3x+5) - \ln(x)(2x-3)}{(x^2-3x+5)^2}$$

trick

trick

$$= \frac{\left(\frac{1}{x}\right)(x^2-3x+5) - \ln(x)(2x-3)x\cdot\left(\frac{1}{x}\right)}{(x^2-3x+5)^2}$$

factor out $\frac{1}{x}$ in front

$$= \frac{\left(\frac{1}{x}\right)\left[(x^2-3x+5) - \ln(x)(2x-3)x\right]}{(x^2-3x+5)^2}$$

move the x to the denominator

$$= \frac{x^2-3x+5 - \ln(x)(2x^2-3x)}{x(x^2-3x+5)^2}$$

[Example 6] (similar to 3.3#69)

$$Let\ \ f(x) = \frac{x}{x^2+4}$$

(a) Find $f'(x)$ and simplify answer.

$$f'(x) = \frac{\left(\frac{d}{dx}x\right)\cdot(x^2+4) - x\,\frac{d}{dx}(x^2+4)}{(x^2+4)^2}$$

quotient rule

warning: Cannot cancel the (x^2+4) here

$$= \frac{(1)(x^2+4) - x(2x)}{(x^2+4)^2}$$

$$= \frac{x^2+4-2x^2}{(x^2+4)^2}$$

$$= \frac{-x^2+4}{(x^2+4)^2}$$

(b) Find the x values where $f'(x) = 0$

$$f'(x) = \frac{-x^2 + 4}{(x^2+4)^2} = 0$$

Recall that a fraction $\frac{a}{b} = 0$ only when $a=0$ and $b \neq 0$!!

Find the values of x that cause numerator $= 0$

$$-x^2 + 4 = 0$$

$$4 = x^2$$

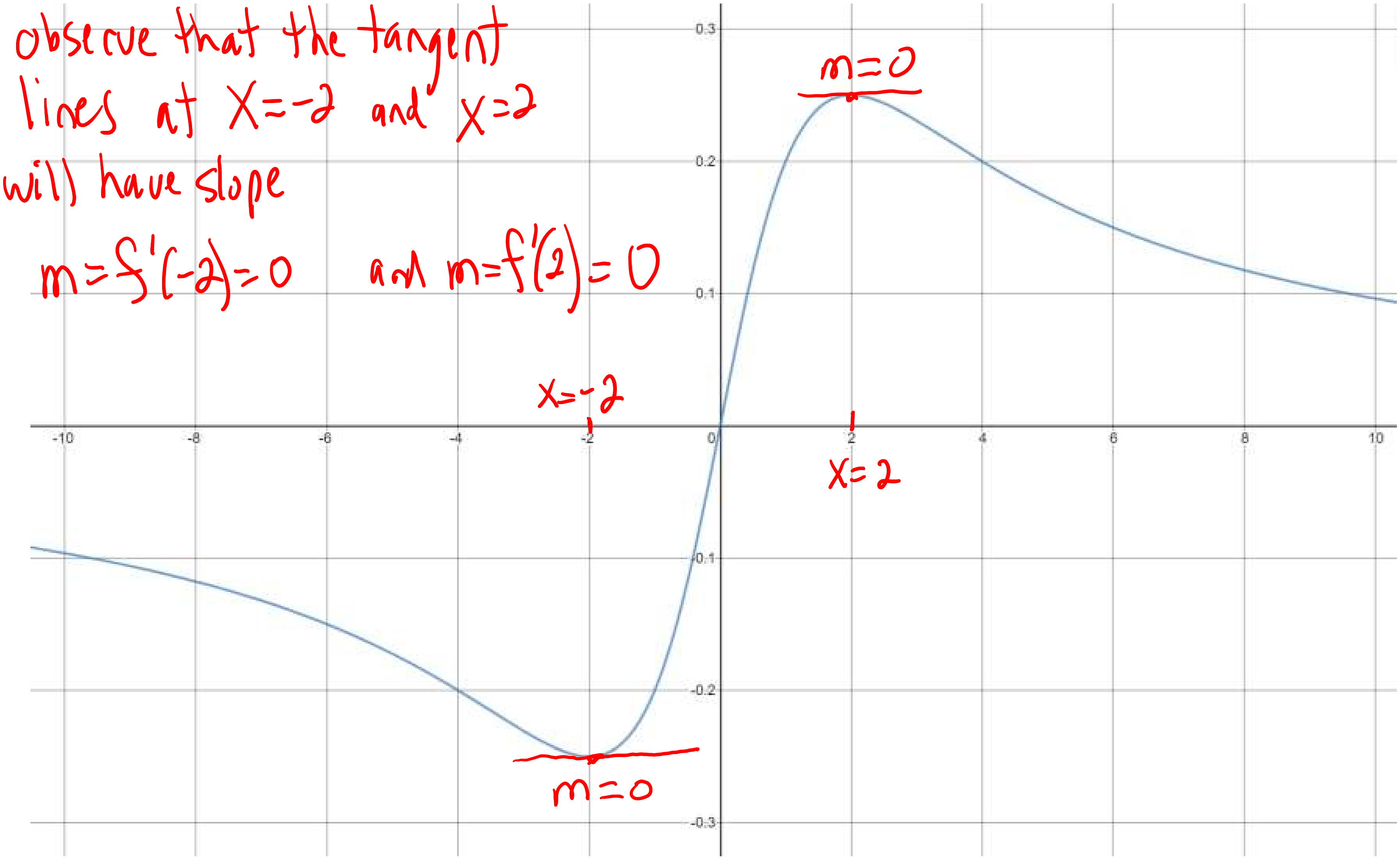
$$x = 2 \quad \text{or} \quad x = -2$$

Check to see if the denominator is non-zero at those x values

- when $x=2$, denominator $= (2^2+4)^2 = (4+4)^2 \neq 0$
- when $x=-2$, denominator $= ((-2)^2+4)^2 = (4+4)^2 \neq 0$

Conclusion: $f'(x) = 0$ at $x=2$ and at $x=-2$ because numerator $= 0$ and denominator $\neq 0$ there.

(c) Illustrate your result of **(b)** on the given graph of $f(x) = \frac{x}{x^2+4}$



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