Subject for this video: Chain Rule Problems with Power Function Outer Function

Reading:

- **General:** Section 3.4: The Chain Rule
- **More Specifically:** Page 204 – middle of page 210, Examples 1,2,3,4A

Homework:

H49: Chain Rule Problems with Power Function Outer Function (3.4#21,27,29,33,37,55,67)

Recall the Derivative Rules that we learned about so far.

Rules from Section 2.5 Basic Differentiation Properties

The Constant Function Rule:	$\dfrac{d}{dx}c = 0$
The Power Rule:	$\dfrac{d}{dx}x^n = nx^{n-1}$
The Sum and Constant Multiple Rule:	$\dfrac{d}{dx}\big(af(x) + bg(x)\big) = a\dfrac{d}{dx}f(x) + b\dfrac{d}{dx}g(x)$

Rules from Section 3.2 Derivatives of Exponential and Logarithmic Functions

Exponential Function Rule #1:	$\frac{d}{dx}e^{(x)} = e^{(x)}$
Exponential Function Rule #2:	$\frac{d}{dx}e^{(kx)} = ke^{(kx)}$
Exponential Function Rule #3:	$\frac{d}{dx}b^{(x)} = b^{(x)} \cdot \ln(b)$
Logarithmic Function Rule #1:	$\frac{d}{dx}\ln(x) = \frac{1}{x}$
Logarithmic Function Rule #2:	$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$

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Rules from Section 3.3 Derivatives of Products and Quotients

The Product Rule:	$\frac{d}{dx} g(x) \cdot h(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$
The Quotient Rule:	$\frac{d}{dx}\left(\frac{top(x)}{bottom(x)}\right) = \frac{top'(x)bottom(x) - top(x)bottom'(x)}{\left(bottom(x)\right)^2}$

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In this video, we will learn how to take the derivative of a *composition of functions*. That is, a function of the form

$$f(x) = outer\big(inner(x)\big)$$

The rule used to find the derivatives of these kinds of functions is called the **Chain Rule**, presented in Section 3.4 of the book.

The Chain Rule

This rule is used for finding the derivative of a *compostion of functions*.

Two equation form:

If

$$f(x) = outer\big(inner(x)\big)$$

then

$$f'(x) = outer'\big(inner(x)\big) \cdot inner'(x)$$

Single equation form:

$$\frac{d}{dx} outer\big(inner(x)\big) = outer'\big(inner(x)\big) \cdot inner'(x)$$

Today: Examples where the *outer function* is a *power function*.

The book solves these using what it calls the *General Power Rule*. That is simply a special case of the *Chain Rule*, and is a completely unnecessary rule. We'll just use the *Chain Rule*.

[Example 1] (similar to 3.4#21) Let $f(x) = 2(3x^4 + 5x^2 + 6)^7$

Find $f'(x)$.

$f'(x) = \frac{d}{dx} 2(3x^4 + 5x^2 + 6)^7$

use constant multiple rule

$= 2 \frac{d}{dx} (3x^4 + 5x^2 + 6)^7$

$= 2 \frac{d}{dx} outer(inner(x))$

use chain rule

$= 2 \cdot outer'(inner(x)) \cdot inner'(x)$

$= 2 \cdot 7(3x^4 + 5x^2 + 6)^6 \cdot (12x^3 + 10x)$

↑ very important parentheses!

$= 14 \cdot (3x^4 + 5x^2 + 6)^6 \cdot (12x^3 + 10x)$

Chain Rule Details

$inner(x) = 3x^4 + 5x^2 + 6$

$outer(\ ) = (\ )^7$ empty version (A power function)

$inner'(x) = \frac{d}{dx}(3x^4 + 5x^2 + 6)$

$= 12x^3 + 10x$

$outer'(\ ) = 7(\ )^6$ empty version

[Example 2] (similar to 3.4#55) Let $f(x) = \dfrac{2}{(3x^4 + 5x^2 + 6)^7}$

Find $f'(x)$.

Start by rewriting $f(x) = \dfrac{2}{(x^4+5x^2+6)^6} = 2\cdot(x^4+5x^2+6)^{-7}$

positive exponent form — power function form

$$f'(x) = \frac{d}{dx}\, 2(x^4+5x^2+6)^{-7}$$

$$= 2\frac{d}{dx}(x^4+5x^2+6)^{-7}$$

$$= 2\frac{d}{dx}\text{outer}(\text{inner}(x))$$

chain rule

$$= 2\,\text{outer}'(\text{inner}(x))\cdot\text{inner}'(x)$$

$$= 2\cdot\frac{-7}{(x^4+5x^2+6)^8}\cdot(12x^3+10x)$$

$$= \frac{-14(12x^3+10x)}{(x^4+5x^2+6)^2} = \frac{-28x(6x^2+5)}{(x^4+5x^2+6)^2}$$

Chain Rule Details

$\text{inner}(x) = x^4+5x^2+6$

$\text{inner}'(x) = 12x^3+10x$

$\text{outer}(\ \ ) = (\ \ )^{-7}$ — Power Function, empty version

$\text{outer}'(\ \ ) = -7(\ \ )^{-7-1}$ — empty version

$= -7(\ \ )^{-8}$

$= -7\cdot\dfrac{1}{(\ \ )^8}$

$= -\dfrac{7}{(\ \ )^8}$ — positive exponent form

[Example 3] (similar to 3.4#67) Let $f(x) = 3\sqrt{x^2 - 3x + 21}$

(A) Find $f'(x)$.

Start by rewriting $f(x) = 3\sqrt{x^2-3x+21} = 3(x^2-3x+21)^{1/2}$

radical form — power function form

$f'(x) = \frac{d}{dx} 3(x^2-3x+12)^{1/2}$

constant multiple rule

$= 3 \frac{d}{dx}(x^2-3x+12)^{1/2}$

$= 3 \frac{d}{dx} \text{outer}(\text{inner}(x))$

chain rule

$= 3\,\text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$

$= 3 \cdot \frac{1}{2\sqrt{(x^2-3x+21)}} \cdot (2x-3)$

$= \frac{3(2x-3)}{2\sqrt{x^2-3x+21}}$

Chain Rule Details

$\text{inner}(x) = x^2 - 3x + 21$

$\text{inner}'(x) = 2x - 3$

$\text{outer}(\ ) = (\ )^{1/2}$ power function, empty version

$\text{outer}'(\ ) = \frac{1}{2}(\ )^{\frac{1}{2}-1}$ power rule

$= \frac{1}{2}(\ )^{-\frac{1}{2}}$ power function form

$= \frac{1}{2} \cdot \frac{1}{(\ )^{1/2}}$

$= \frac{1}{2(\ )^{1/2}}$ positive exponent form

$= \frac{1}{2\sqrt{(\ )}}$ radical form

(B) Find the equation of the line tangent to the graph of $f(x)$ at $x = 4$.

We need to build the equation $(y - f(a)) = f'(a)(x - a)$ — Point slope form of the equation of the tangent line

Get Parts

$a = 4$ — the x coordinate of the point of tangency

$f(a) = f(4) = 3\sqrt{(4)^2 - 3(4) + 21} = 3\sqrt{16 - 12 + 21} = 3\sqrt{25} = 3 \cdot 5 = 15$ — y coordinate of the point of tangency

Substitute $x = 4$ into $f(x) = 3\sqrt{x^2 - 3x + 21}$

$f'(a) = f'(4) = \dfrac{3(2(4) - 3)}{2\sqrt{(4)^2 - 3(4) + 21}} = \dfrac{3(8-3)}{2 \cdot 5} = \dfrac{3 \cdot 5}{2 \cdot 5} = \dfrac{3}{2}$ — slope of the tangent line

Substitute $x = 4$ into $f'(x) = \dfrac{3(2x-3)}{2\sqrt{x^2 - 3x + 21}}$

Substitute parts into the equation

$$(y - 15) = \frac{3}{2}(x - 4)$$

point slope form of the equation of the tangent line

Convert to slope intercept form

$y - 15 = \left(\frac{3}{2}\right)(x - 4) = \left(\frac{3}{2}\right)x - \left(\frac{3}{2}\right)4 = \left(\frac{3}{2}\right)x - 6$

$$y = \left(\frac{3}{2}\right)x + 9$$

Slope-intercept form of the equation of the tangent line

(C) Find x coordinates of all points on the graph of $f(x)$ that have horizontal tangent lines.

Important connection: Horizontal tangent lines have slope $m=0$

Also: The slope of the line tangent to graph of $f(x)$ at $x=c$ $= f'(c)$

We are looking for all $x=c$ such that $f'(c)=0$

Strategy: Set $f'(x)=0$ and solve for x

$$0 = f'(x) = \frac{3(2x-3)}{2\sqrt{x^2-3x+21}}$$

Recall that a fraction $\frac{a}{b}=0$ only when $a=0$ and $b \neq 0$

Find all x values where the numerator $=0$

$$0 = 3(2x-3)$$

$$0 = 2x-3$$

$$x = \frac{3}{2}$$

horizontal tangent line at $x=\frac{3}{2}$ because $f'(x)=0$ there

Check the denominator to see if it is non-zero

$$2\sqrt{\left(\tfrac{3}{2}\right)^2 - 3\left(\tfrac{3}{2}\right) + 21} = 2\sqrt{\tfrac{9}{4} - \tfrac{9}{2} + 21} = 2\sqrt{-\tfrac{9}{4}+21} = 2\sqrt{\tfrac{75}{4}} \neq 0$$

(D) Illustrate the results from **(B)** and **(C)** on the given graph of $f(x)$.