Subject for this video:

Chain Rule Problems with Exponential or Logarithmic Outer Function

Reading:

- **General:** Section 3.4: The Chain Rule
- **More Specifically:** middle of page 210 – middle of page 212, Examples 4BC,5,6

Homework:

H50: Chain Rule Problems with Exponential or Logarithmic Outer Function (3.4#25,39,40)

Derivative Rules from Section 2.5 Basic Differentiation Properties

The Constant Function Rule:	$\frac{d}{dx}c = 0$
The Power Rule:	$\frac{d}{dx}x^n = nx^{n-1}$
The Sum and Constant Multiple Rule:	$\frac{d}{dx}\big(af(x) + bg(x)\big) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$

Derivative Rules from Section 3.2 Derivatives of Exponential and Logarithmic Functions

Exponential Function Rule #1:	$\frac{d}{dx}e^{(x)} = e^{(x)}$
Exponential Function Rule #2:	$\frac{d}{dx}e^{(kx)} = ke^{(kx)}$
Exponential Function Rule #3:	$\frac{d}{dx}b^{(x)} = b^{(x)} \cdot \ln(b)$
Logarithmic Function Rule #1:	$\frac{d}{dx}\ln(x) = \frac{1}{x}$
Logarithmic Function Rule #2:	$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$

Derivative Rules from Section 3.3 Derivatives of Products and Quotients

The Product Rule:	$\frac{d}{dx} g(x) \cdot h(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$
The Quotient Rule:	$\frac{d}{dx}\left(\frac{top(x)}{bottom(x)}\right) = \frac{top'(x)bottom(x) - top(x)bottom'(x)}{\left(bottom(x)\right)^2}$

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Derivative Rule from Section 3.4 The Chain Rule

The Chain Rule:	$\frac{d}{dx} outer\left(inner(x)\right) = outer'\left(inner(x)\right) \cdot inner'(x)$

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Today: Examples where the *outer function* is an *exponential* or *logarithmic function*

[Example 1] (similar to 3.4#25) Let $f(x) = e^{(kx)}$

Find $f'(x)$ using the *Chain Rule*

$$f'(x) = \frac{d}{dx} e^{(kx)}$$

$$= \frac{d}{dx} \text{outer}(\text{inner}(x))$$

chain rule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= e^{(kx)} \cdot k$$

$$= ke^{(kx)}$$

Chain Rule Details

$\text{inner}(x) = kx$

$\text{inner}'(x) = \frac{d}{dx} kx = k$

$\text{outer}(\ ) = e^{(\ )}$ exponential function, empty version

$\text{outer}'(\ ) = e^{(\ )}$ empty version

Recall the Exponential Function Rule #2, introduced in the video for Homework H40.

Exponential Function Rule #2: $\frac{d}{dx}e^{(kx)} = ke^{(kx)}$

(The book does not present this equation as a derivative rule. That's unfortunate, because it is very useful. That's why I call it a *derivative rule* in these notes.)

This derivative can be found by using the *Definition of the Derivative*.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{(k(x+h))} - e^{(kx)}}{h}$$

But a key step in the calculation uses a mathematical fact that is above the level of this course. If we wanted to compute the derivative using the *Definition of the Derivative*, we would just have to accept that fact as given. For that reason, we did not discuss the calculation of this derivative in the video for Homework H40, and we accepted the Exponential Function Rule #2 without proof.

We now see that the derivative can also be found by using the *Chain Rule*.

[Example 2] (similar to 3.4#39) Let $f(x) = e^{(-x^2+4x-4)}$

(A) Find $f'(x)$

$$f'(x) = \frac{d}{dx} e^{(-x^2+4x-4)}$$

$$= \frac{d}{dx} \text{outer}(\text{inner}(x))$$

chain rule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= e^{(-x^2+4x+4)} \cdot (-2)(x-2)$$

$$= -2(x-2)e^{(-x^2+4x+4)}$$

Chain Rule Details

$\text{inner}(x) = -x^2+4x-4$

$\text{inner}'(x) = -2x+4$

$= 2(-x+2)$

$= -2(x-2)$

$\text{outer}(\ ) = e^{(\ )}$ exponential function empty version

$\text{outer}'(\ ) = e^{(\ )}$

(B) Find the equation of the line tangent to the graph of $f(x)$ at $x = 0$.

we need to build $(y - f(a)) = f'(a) \cdot (x - a)$ Point slope form of the equation of the tangent line

Get parts

$a = 0$ the x coordinate of the point of tangency

$f(a) = f(0) = e^{(-(0)^2 + 4(0) - 4)} = e^{(-4)} = \frac{1}{e^4}$ y coord of the point of tangency

sub x=0 into formula for f(x)

(positive exponent form)

$f'(a) = f'(0) = -2((0)-2)e^{(-(0)^2+4(0)-4)} = 4e^{-4} = \frac{4}{e^4}$ Slope of the tangent line

sub x=0 into formula for f'(x)

Substitute Parts into the Tangent Line Equation

$(y - \frac{1}{e^4}) = \frac{4}{e^4} \cdot (x - 0)$ Point slope form of the equation of the tangent line.

Convert to slope intercept form

$y - \frac{1}{e^4} = \left(\frac{4}{e^4}\right)x$

$y = \left(\frac{4}{e^4}\right)x + \frac{1}{e^4}$ Slope intercept form of the equation of the tangent line.

(C) Find x coordinates of all points on the graph of $f(x)$ that have horizontal tangent lines.

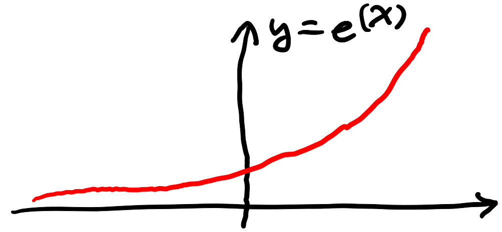
Strategy: set $f'(x)=0$ and solve for x

$$0 = f'(x) = -2(x-2)e^{(-x^2+4x-4)}$$

this is of the form $0 = a \cdot b$

The Zero Product Property tells us that either $a=0$ or $b=0$ or both

Recall that $e^{(\text{anything})} > 0$



all $y > 0$

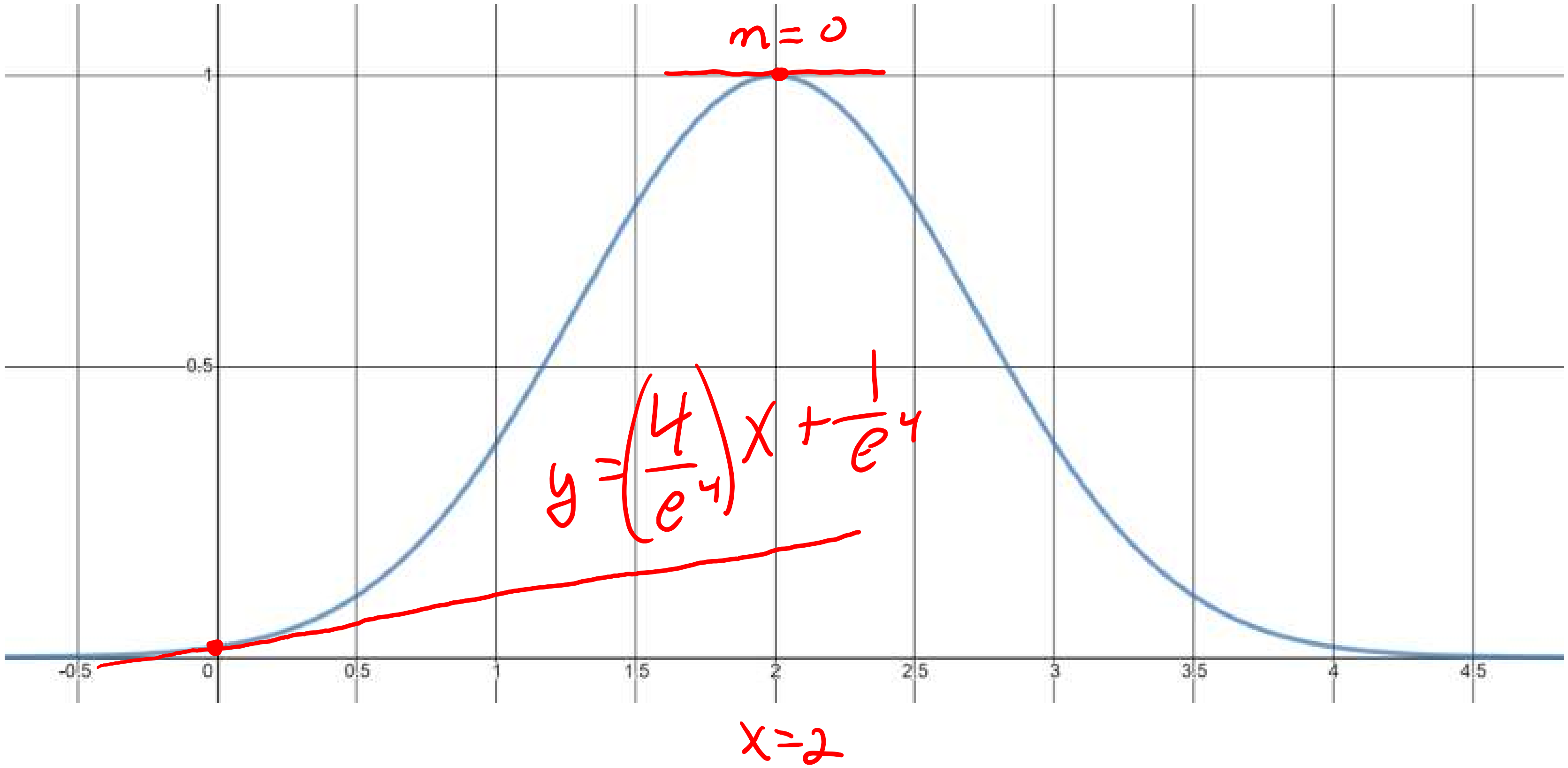
So $e^{(-x^2+4x-4)}$ is never 0.

Therefore, it must be that $-2(x-2) = 0$

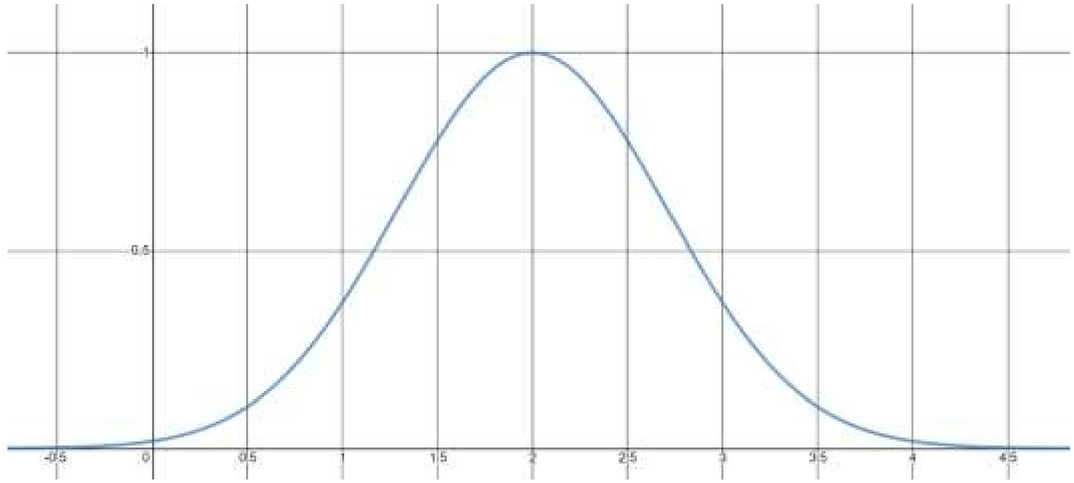
This is only true when $x = 2$

Horizontal tangent line there

(D) Illustrate the results from **(B)** and **(C)** on the given graph of $f(x)$.



Bell-Shaped Curves



Notice that the graph of $f(x) = e^{(-x^2+4x-4)}$ looks kind of like a bell. The shape is common in math and is called a ***bell-shaped curve***.

Curves of that general shape can be produced by a variety of types of functions. Perhaps the most common type is

$$f(x) = e^{(polynomial)}$$

where the polynomial has degree 2 and a negative leading coefficient. Notice that the function

$$f(x) = e^{(-x^2+4x-4)}$$

is of that type.

Example where the outer function is a logarithmic function

[Example 3] (similar to 3.4#40) Let $f(x) = \ln(2x^4 - x^2 + 2)$

(A) Find $f'(x)$

$f'(x) = \frac{d}{dx} \ln(2x^4 - x^2 + 2)$

$= \frac{d}{dx} \text{outer}(\text{inner}(x))$

chain rule

$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$

$= \frac{1}{(2x^4 - x^2 + 2)} (8x^3 - 2x)$

$= \frac{8x^3 - 2x}{2x^4 - x^2 + 2}$

Chain Rule Details

$\text{inner}(x) = 2x^4 - x^2 + 2$

$\text{inner}'(x) = 8x^3 - 2x$

$\text{outer}(\ \ ) = \ln(\ \ )$ logarithmic function empty version

$\text{outer}'(\ \ ) = \frac{1}{(\ \ )}$ empty version

(B) Find the equation of the line tangent to the graph of $f(x)$ at $x = 1$.

We need to build $(y - f(a)) = f'(a)(x - a)$

Get Parts

$a = 1$

$f(a) = f(1) = \ln(2(1)^4 - (1)^2 + 2) = \ln(3)$

↑ sub $x=1$ into $f(x) = \ln(2x^4 - x^2 + 2)$

$f'(a) = f'(1) = \dfrac{8(1)^3 - 2(1)}{2(1)^4 - 2(1)^2 + 2} = \dfrac{6}{3} = 2$

↑ Sub $x=1$ into $f'(x) = \dfrac{8x^3 - 2x}{2x^4 - x^2 + 2}$

Substitute parts into the equation

$(y - \ln(3)) = 2(x - 1)$

Convert to slope intercept form

$y - \ln(3) = 2(x-1) = 2x - 2$

$y = 2x - 2 + \ln(3)$

(C) Find x coordinates of all points on the graph of $f(x)$ that have horizontal tangent lines.

Strategy: Set $f'(x) = 0$ and solve for x

$$0 = f'(x) = \frac{8x^3 - 2x}{2x^4 - x^2 + 2}$$

Remember that $\frac{a}{b} = 0$ only when $a = 0$ and $b \neq 0$

Start by finding x values that cause numerator $= 0$

$$0 = 8x^3 - 2x = \underbrace{2x}\cdot 4x^2 - \underbrace{2x}\cdot 1 = \underbrace{2x}\underbrace{(4x^2 - 1)}$$

difference of two squares

$$= 2x(2x+1)(2x-1)$$

Solutions are: $x = 0$

$2x + 1 = 0$

$2x = -1$

$x = -\frac{1}{2}$

$2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$

$x = -\frac{1}{2}, 0, \frac{1}{2}$

(D) Illustrate the results from **(B)** and **(C)** on the given graph of $f(x)$.