Subject for this video:

Applied Problems Involving the Chain Rule

Reading:

- **General:** Section 3.4: The Chain Rule
- **More Specifically:** There is no discussion of, or examples of, applied problems involving the Chain Rule in Section 3.4. But discussion or book examples are not really needed: We have seen applied problems in book sections 2.4, 2.5, 3.2, and 3.3. All the applied problems involve the same general concepts. Only the particular technique used to find the derivative varies.

Homework:

H51: Applied Problems Involving the Chain Rule (3.4#91,95)

Recall this definition of *Average Rate of Change*, from the video for Homework H25:

Definition of *Average Rate of Change*

words: *the* ***average rate of change*** *of f from $x = a$ to $x = b$*

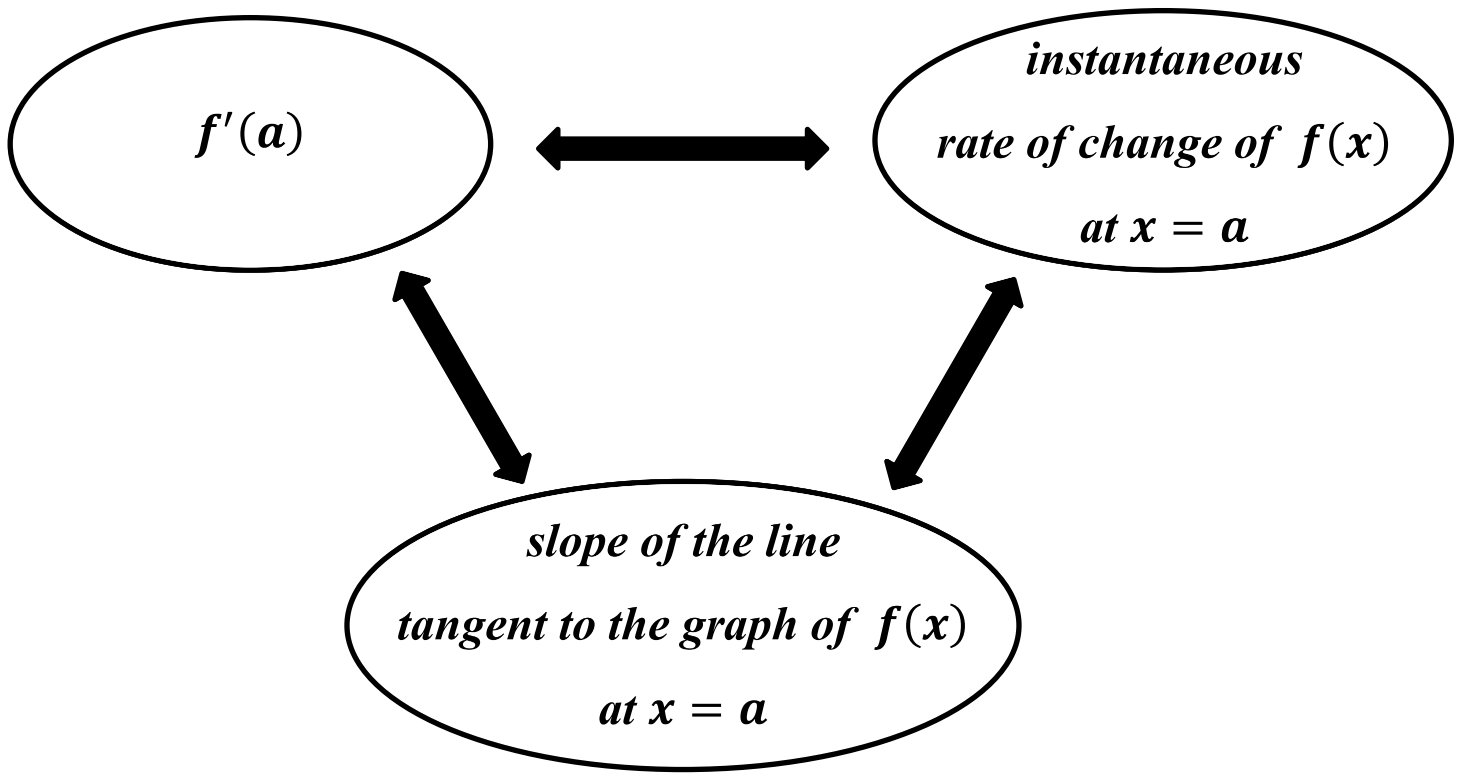
usage: f is a function that is continuous on the interval $[a, b]$.

meaning: the number $m = \frac{f(b)-f(a)}{b-a}$

graphical interpretation: The number m is the slope of the ***secant line*** that touches the graph of f at the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$.

remark: The average rate of change m is a number.

And recall this diagram showing *Three Equal Quantities*, from the video for Homework H33:



Three Equal Quantities

(the most important concept of the second month of the course)

Derivative Rules from Section 2.5 Basic Differentiation Properties

The Constant Function Rule:	$\frac{d}{dx}c = 0$
The Power Rule:	$\frac{d}{dx}x^n = nx^{n-1}$
The Sum and Constant Multiple Rule:	$\frac{d}{dx}\big(af(x) + bg(x)\big) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$

Derivative Rules from Section 3.2 Derivatives of Exponential and Logarithmic Functions

Exponential Function Rule #1:	$\frac{d}{dx}e^{(x)} = e^{(x)}$
Exponential Function Rule #2:	$\frac{d}{dx}e^{(kx)} = ke^{(kx)}$
Exponential Function Rule #3:	$\frac{d}{dx}b^{(x)} = b^{(x)} \cdot \ln(b)$
Logarithmic Function Rule #1:	$\frac{d}{dx}\ln(x) = \frac{1}{x}$
Logarithmic Function Rule #2:	$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$

Derivative Rules from Section 3.3 Derivatives of Products and Quotients

The Product Rule:	$\frac{d}{dx}g(x)\cdot h(x)=g'(x)\cdot h(x)+g(x)\cdot h'(x)$
The Quotient Rule:	$\frac{d}{dx}\left(\frac{top(x)}{bottom(x)}\right)=\frac{top'(x)bottom(x)-top(x)bottom'(x)}{\big(bottom(x)\big)^2}$

.

Derivative Rule from Section 3.4 The Chain Rule

The Chain Rule:	$\frac{d}{dx}outer\big(inner(x)\big)=outer'\big(inner(x)\big)\cdot inner'(x)$

.

[Example 1] (similar to 3.4#91) A company makes ventilators. The daily cost function is

$$C(x) = 60 + 40\sqrt{4x + 36}$$

In this equation,

x is the number of ventilators produced each day.

$C(x)$ is the cost per day, in thousands of dollars.

(A) Find $C(0)$ (exact answer)

$$C(0) = 60 + 40\sqrt{4(0) + 36} = 60 + 40\sqrt{36} = 60 + 40 \cdot 6$$
$$= 60 + 240 = 300$$

(B) Find $C(16)$ and $C(91)$ (exact answers)

$$C(16) = 60 + 40\sqrt{4(16) + 36} = 60 + 40\sqrt{64 + 36} =$$
$$= 60 + 40\sqrt{100} = 60 + 40 \cdot 10 = 460$$
$$C(91) = 60 + 40\sqrt{4(91) + 36} = 60 + 40\sqrt{364 + 36} =$$
$$= 60 + 40\sqrt{400} = 60 + 40 \cdot 20 = 860$$

(C) Find $C'(16)$ and $C'(91)$

Need to get $c'(x)$ first

Start by rewriting $C(x) = 60 + 40\sqrt{4x+36} = 60 + 40(4x+36)^{1/2}$

radical form — power function form

$$c'(x) = \frac{d}{dx}\left(60 + 40(4x+36)^{1/2}\right)$$

$$= \frac{d}{dx}60 + 40\frac{d}{dx}(4x+36)^{1/2}$$

$$= 0 + 40\frac{d}{dx}\,\text{outer}(\text{inner}(x))$$

chain rule

$$= 40 \cdot \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= 40 \cdot \frac{1}{\cancel{2}\sqrt{(4x+36)}} \cdot \cancel{4} = \frac{80}{\sqrt{4x+36}}$$

Chain rule details

$\text{inner}(x) = 4x+36$

$\text{inner}'(x) = \frac{d}{dx}(4x+36) = 4$

$\text{outer}(\ ) = (\ )^{1/2}$ empty version

$\text{outer}'(\ ) = \frac{1}{2}(\ )^{\frac{1}{2}-1}$

$= \frac{1}{2}(\ )^{-\frac{1}{2}}$ empty version

$= \frac{1}{2} \cdot \frac{1}{(\ )^{1/2}}$

$= \frac{1}{2\sqrt{(\ )}}$

$$c'(16) = \frac{80}{\sqrt{4(16)+36}} = \frac{80}{\sqrt{100}} = \frac{80}{10} = 8$$

$$c'(91) = \frac{80}{\sqrt{4(91)+36}} = \frac{80}{\sqrt{400}} = \frac{80}{20} = 4$$

(D) Interpret the results of **(A), (B),** and **(C)**.

(A) The fixed costs are 300 thousand dollars per day.

(B) If the company produces 16 ventilators per day, their costs will be 460 thousand dollars per day

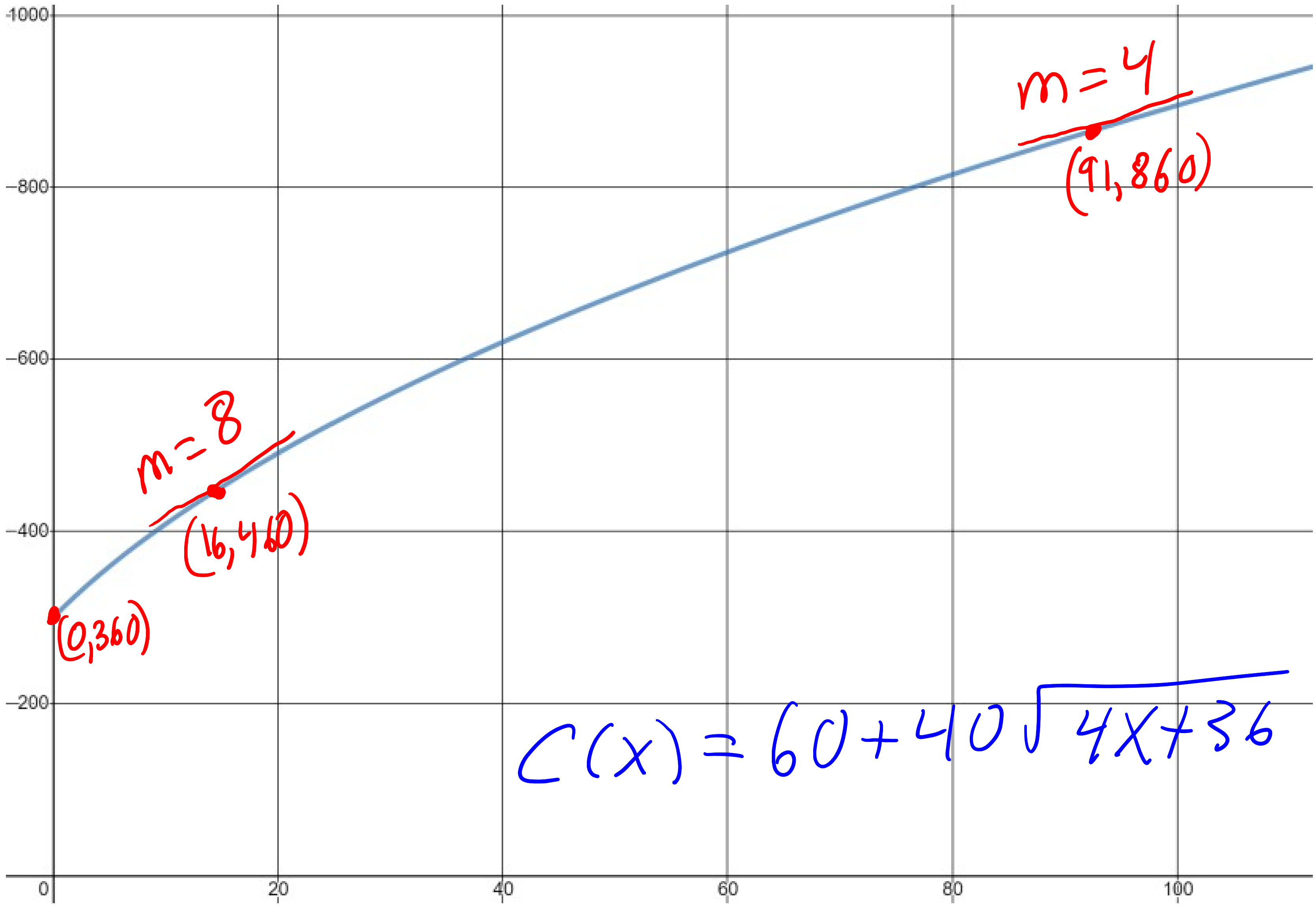
If the company produces 91 ventilators per day, their costs will be 860 thousand dollars per day.

(C) If the production changes from 16 ventilators per day to 17, the costs will increase by roughly 8 thousand dollars "The cost of the next ventilator"

If the production changes from 91 ventilators per day to 92, the costs will increase by roughly 4 thousand dollars. "the cost of the next ventilator"

Notice: the "cost of the next ventilator" decreases

(E) Illustrate each of the quantities found in **(A)**, **(B)**, **(C)** on the given graph of the cost function.



[Example 2] (similar to 3.4#95) A drug is administered by injection.

The drug concentration (in milligrams per liter) in the bloodstream t hours after the injection is given by the formula

$$c(t) = 3e^{\left(-\frac{t}{2}\right)}\ for\ 0 \leq t$$

(A) Find the concentration after 1 hour and after 4 hours. (Give an exact answers in symbols and approximate answers in decimals. Include units in your answer.)

$$c(1) = 3e^{\left(-\frac{1}{2}\right)} = 3 \cdot \frac{1}{e^{1/2}} = \underbrace{\frac{3}{\sqrt{e}}}_{\text{exact}} \approx \underbrace{1.82}_{\text{approximation}} \text{ milligrams per liter}$$

$$c(4) = 3e^{\left(-\frac{4}{2}\right)} = 3 \cdot e^{-2} = 3 \cdot \frac{1}{e^2} = \underbrace{\frac{3}{e^2}}_{\text{exact}} \approx \underbrace{0.41}_{\text{approximation}} \text{ milligrams per liter}$$

(B) Find the rate of change of the concentration after 1 hour. (Give an exact answer in symbols and then approximate answer in decimals. Include units in your answer.)

We are being asked for $C'(1)$

Strategy:
- find $c'(t)$
- substitute in $t=1$ to get $c'(1)$

$$c'(t) = \frac{d}{dt}\left(3e^{\left(-\frac{t}{2}\right)}\right) = 3\frac{d}{dt}e^{\left(-\frac{1}{2}\cdot t\right)}$$

$$-\frac{t}{2} = -\frac{1}{2}\cdot t$$

constant multiple rule

$$= 3\left(-\frac{1}{2}\right)e^{\left(\left(-\frac{1}{2}\right)\cdot t\right)} = -\frac{3}{2}e^{\left(-\frac{t}{2}\right)}$$

exponential function Rule#2

$$c'(1) = -\frac{3}{2}e^{\left(-\frac{1}{2}\right)} = -\frac{3}{2}\cdot\frac{1}{e^{1/2}} = -\frac{3}{2\sqrt{e}} \approx -0.91 \text{ milligrams per liter per hour}$$

exact

approximate

Concentration is decreasing at a rate of 0.91 milligrams per liter per hour

(C) Find the rate of change of the concentration after 4 hours. (Give an exact answer in symbols and then approximate answer in decimals. Include units in your answer.)

$$c'(4) = -\frac{3}{2}e^{\left(-\frac{4}{2}\right)} = -\frac{3}{2}e^{(-2)} = -\frac{3}{2}\frac{1}{e^2}$$

Substitute $t=4$ into $c'(t) = -\frac{3}{2}e^{-\frac{t}{2}}$

$$= -\frac{3}{2e^2} \approx -.20 \text{ milligrams per liter per hour}$$

exact

approximate

At time 4 hours, concentration is decreasing at a rate roughly 0.20 milligrams per liter per hour

(D) Find the average rate of change of the concentration from 1 to 4 hours. (Give an exact answer in symbols and then approximate answer in decimals. Include units in your answer.)

$$\text{Average rate of change} = m = \frac{c(4) - c(1)}{4 - 1} = \frac{c(4) - c(1)}{3}$$

$$= \frac{\frac{3}{e^2} - \frac{3}{\sqrt{e}}}{3} = \frac{1}{3}\left(\frac{3}{e^2} - \frac{3}{\sqrt{e}}\right)$$

$$= \frac{1}{e^2} - \frac{1}{\sqrt{e}} \approx -.41 \quad \text{milligrams per liter per hour}$$

exact approximate

Over the 3 hour period from $t=1$ to $t=4$, the concentration decreased at an average rate of 0.41 milligrams per liter per hour

(E) Illustrate the quantities found in **(A)**, **(B)**, **(C)**, **(D)** on the given graph of the concentration.