Subject for this video:

Local Extrema and the First Derivative Test

Reading:

- **General:** Section 4.1 First Derivative and Graphs
- **More Specifically:** In my opinion, Section 4.1 is not organized very well. The topics do not progress from simple to more complex. The exercises also do not progress from simple to more complex. Plus, the ordering of the exercises does not match the order of presentation in the reading. You may find the book a little frustrating to read. I have chosen to present concepts from Section 4.1 in an order that I feel does progress from simple to more complex. It is not possible to give guidance about what parts of Section 4.1, what examples, correspond to the topics in this video, because the topics here are scattered throughout Section 4.1

Homework:

H56: The First Derivative Test (4.1#17,43,45,77, 85,97)

Useful Section 4.1 Concepts Discussed in Videos for H53, H54, H55

Correspondence between

sign behavior of $f'(x)$ at a particular $x = c$ and behavior of the graph of $f(x)$ at $x = c$

- If $f'(c)$ is positive then the line tangent to graph of $f(x)$ at $x = c$ tilts upward
- If $f'(c)$ is negative then the line tangent to graph of $f(x)$ at $x = c$ tilts downward
- If $f'(c)$ is negative then the line tangent to graph of $f(x)$ at $x = c$ is horizontal

Correspondence between

sign behavior of $f'(x)$ on an interval (a, b) and behavior of graph of $f(x)$ on the interval (a, b)

- If $f'(x)$ is positive on an interval (a, b) then $f(x)$ is increasing on the interval (a, b).
- If $f'(x)$ is negative on an interval (a, b) then $f(x)$ is decreasing on the interval (a, b).
- If $f'(x)$ is zero on an interval (a, b) then $f(x)$ is constant on the interval (a, b).

Definition of *Partition Number for* $f'(x)$

Words: *partition number for* $f'(x)$

Meaning: a number $x = c$ such that $f'(c) = 0$ or $f'(c)$ *does not exist*

Definition of *Critical Number for* $f(x)$

Words: *critical number for* $f(x)$

Meaning: a number $x = c$ that satisfies these two requirements:

- The number $x = c$ is a partition number for $f'(x)$.
- The number $x = c$ is in the domain of $f(x)$.

That is,

- $f'(c) = 0$ or $f'(c)$ does not exist
- $f(c)$ exists

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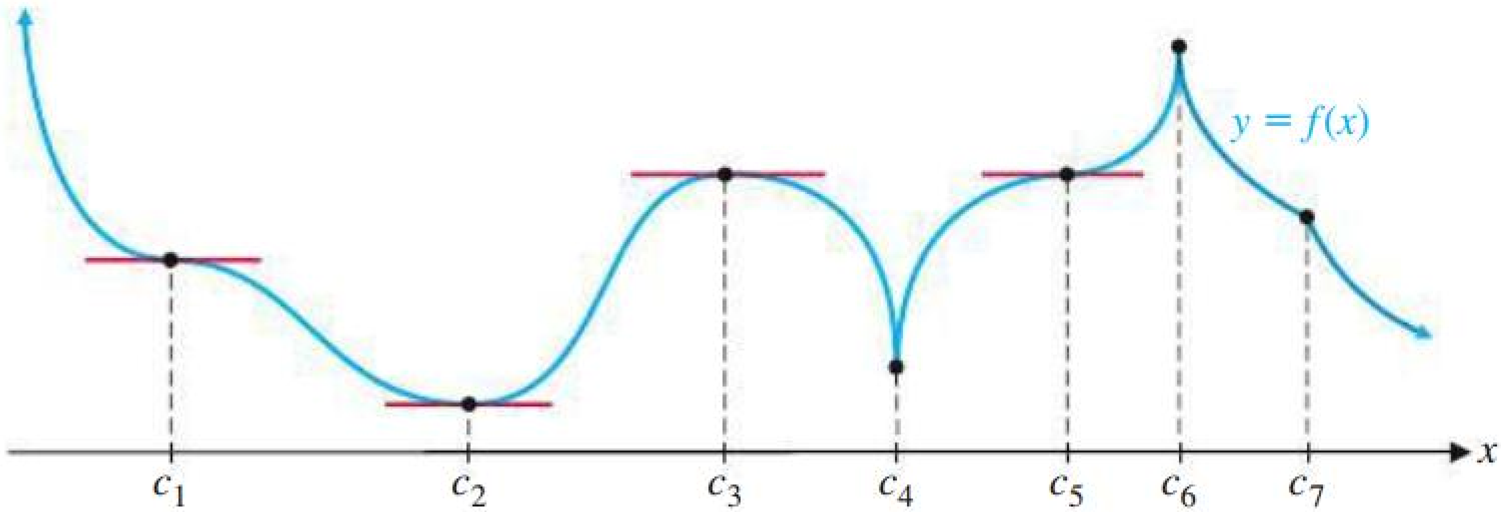
Local Extrema

When a graph of a function is available, it is easy to notice high and low points on it.



We call the y coordinates of such a point a *local maximum* or a *local minimum*. The definitions follow on the next page.

Definition of *Local Maximum*

Words: *a **local maximum** for* $f(x)$.

Meaning: a y value $y = f(c)$ such that

- $f(x)$ is continuous on an interval (m, n) containing $x = c$
- The y value $f(c)$ is the *greatest* y value on the interval (a, b).

  That is, $f(c) \geq f(x)$ for all x in the interval (m, n).

Definition of *Local Minimum*

Words: *The y value $f(c)$ is a **local minimum** for* $f(x)$.

Meaning: a y value $y = f(c)$ such that

- $f(x)$ is continuous on an interval (m, n) containing $x = c$
- The y value $f(c)$ is the *least* y value on the interval (a, b).

  That is, $f(c) \leq f(x)$ for all x in the interval (m, n).

Definition of *Local Extremum*

Words: *a **local extremum** for* $f(x)$.

Meaning: a y value $y = f(c)$ that is a ***local maximum*** or a ***local minimum***

What if a function $f(x)$ is given by a formula, and not by a graph. Is there some way to scrutinize the *formula* for $f(x)$ and determine the *local extrema*?

It turns out that there is a way.

The key is to notice something about the sample graph shown earlier: The high and low points always occur at points on the graph of $f(x)$ that have either a horizontal tangent line, or no tangent line (because there is a cusp on the graph). In other words,

THEOREM 2 Local Extrema and Critical Numbers

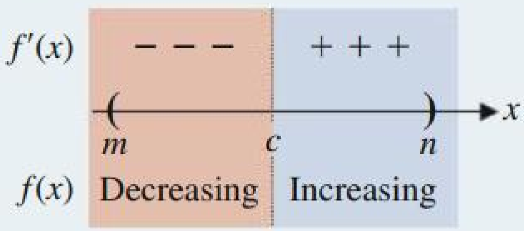
If $f(c)$ is a local extremum of the function f, then c is a critical number of f.

Furthermore, notice that not all critical numbers for $f(x)$ correspond to points on the graph that have a local max or min. The key criterion is that for a critical number $x = c$ to be the location of a local max or min, $f(x)$ must change from increasing to decreasing, or from decreasing to increasing, at $x = c$. That is the essence of the *First-Derivative Test*.

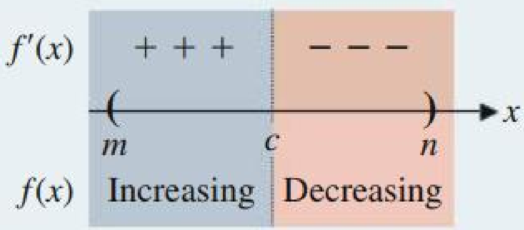
PROCEDURE First-Derivative Test for Local Extrema

Let c be a critical number of f $[f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ is not defined$]$. Construct a sign chart for $f'(x)$ close to and on either side of c.

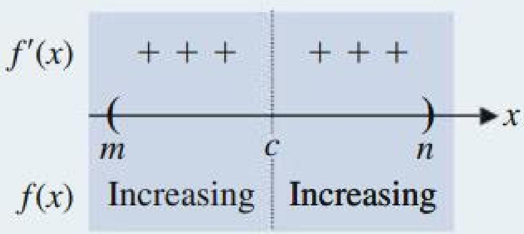
Sign Chart | ***f*(*c*)**



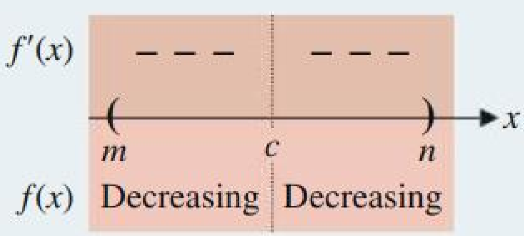
$f(c)$ is a local minimum.
If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a local minimum.



$f(c)$ is a local maximum.
If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a local maximum.



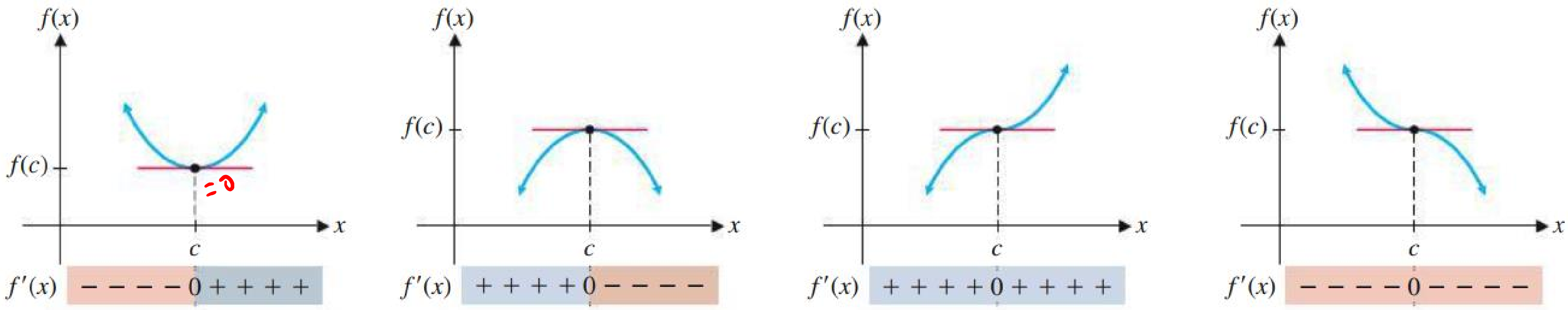
$f(c)$ is not a local extremum.
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.



$f(c)$ is not a local extremum.
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.

It is useful to look at sample graphs with the First Derivative Test in mind.

$f'(c) = 0$: Horizontal tangent



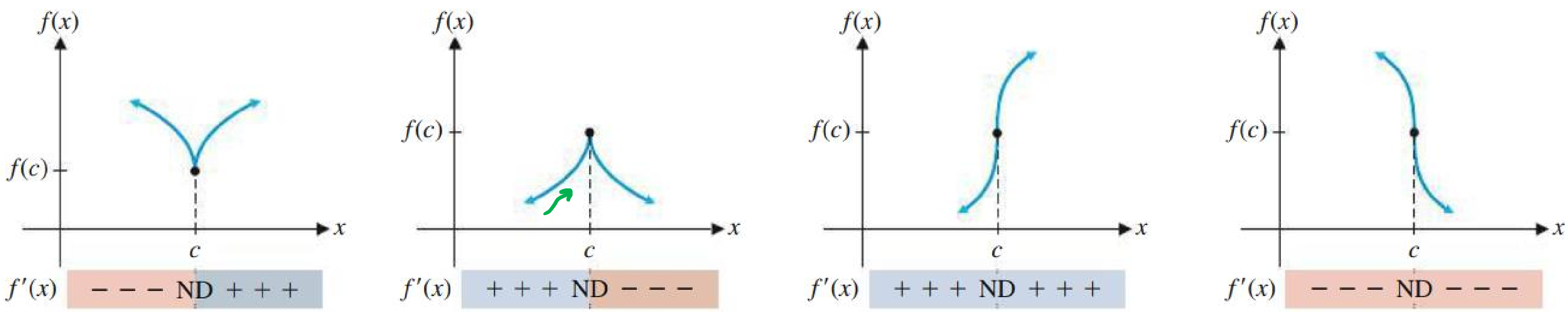
(A) $f(c)$ is a local minimum

(B) $f(c)$ is a local maximum

(C) $f(c)$ is neither a local maximum nor a local minimum

(D) $f(c)$ is neither a local maximum nor a local minimum

$f'(c)$ is not defined but $f(c)$ is defined



(E) $f(c)$ is a local minimum

(F) $f(c)$ is a local maximum

(G) $f(c)$ is neither a local maximum nor a local minimum

(H) $f(c)$ is neither a local maximum nor a local minimum

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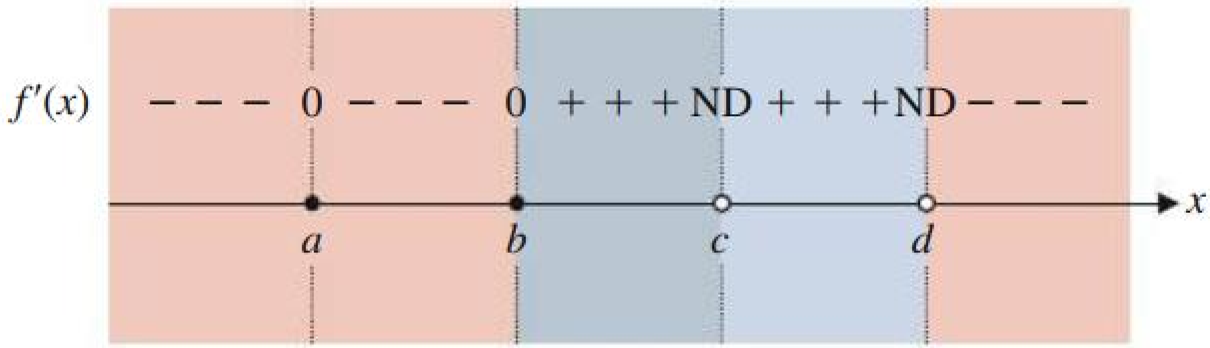
[Example 1] (Similar to 4.1#17) A function $f(x)$ is continuous on the interval $(-\infty, \infty)$.

The sign chart for $f'(x)$ is shown below.



Find the x coordinates of all local extrema of $f(x)$.

Solution

Local extrema can only occur at x values that are critical numbers of $f(x)$.

- $f(c)$ must exist always true because $f(x)$ continuous
- $f'(c) = 0$ or $f'(c)$ does not exist.

only happens at $x = a, b, c, d$

Local min at $x = b$ because $f'(x)$ changes from neg to pos.

Local max at $x = d$ because $f'(x)$ changes from pos to neg.

[Example 2] (similar to 4.1#49,51,53,55) Revisit **[Example 2]** from the Video for H54:

$$f(x) = -x^4 + 4x^3$$

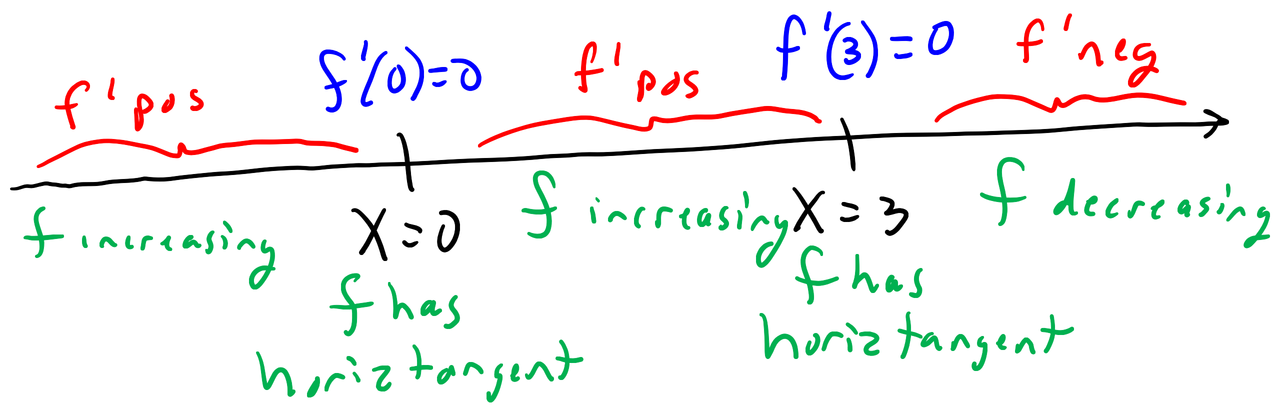
(A) Find the local extrema of $f(x)$.

Solution In the earlier video, we found

$$f'(x) = -4x^3 + 12x^2 = -4x^2(x-3)$$

Critical numbers $x=0$, $x=3$

Sign chart for $f'(x)$



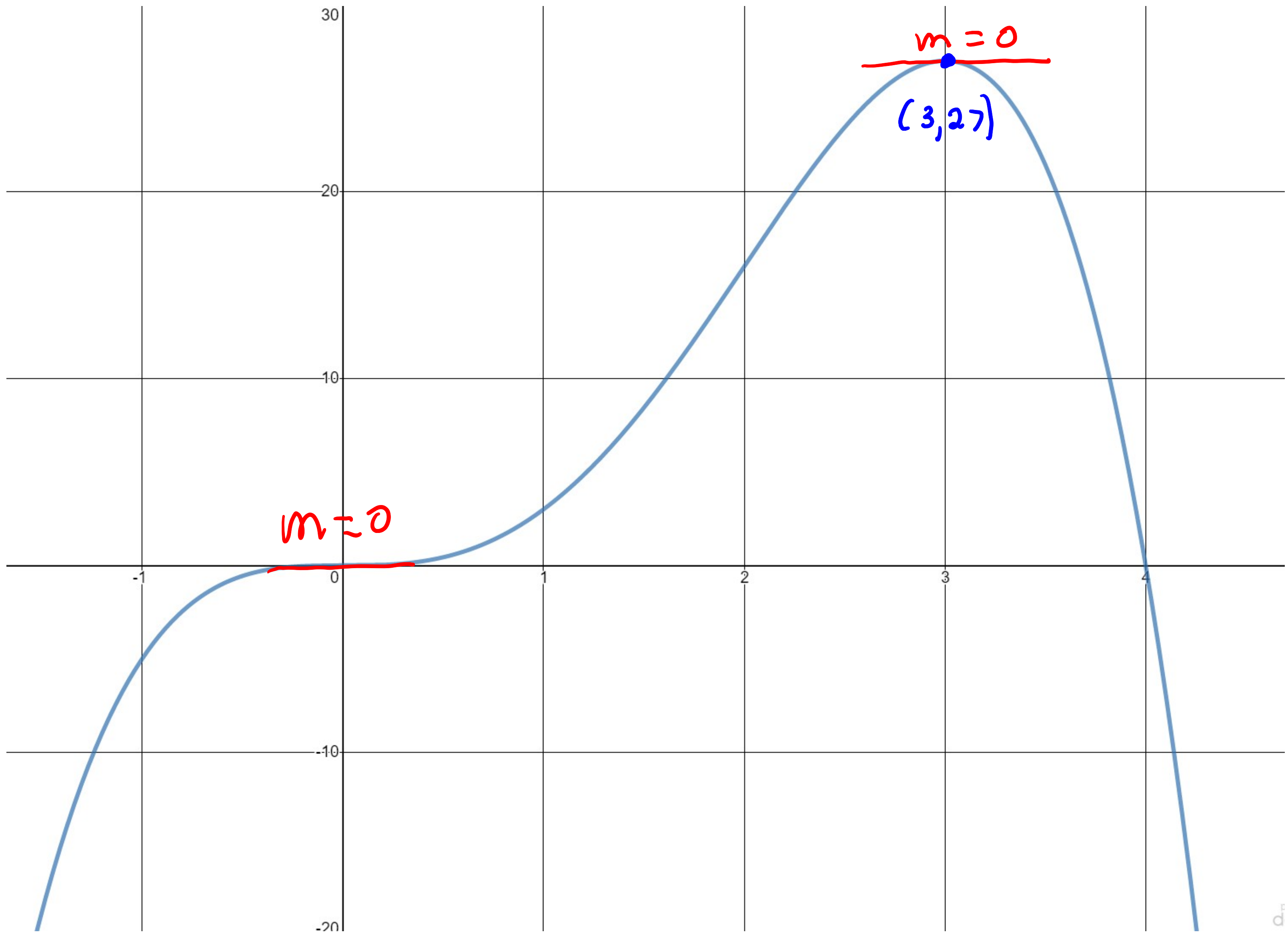
f has local max at $x=3$ because $f'(x)$ changes from pos to neg.

The value of the local max is $y = f(3)$

$$f(3) = -(3)^4 + 4(3)^3 = -81 + 4(27) = -81 + 108 = 27$$

The local max is $y = 27$. That is the local max is $f(3) = 27$

(B) Illustrate on the given graph of $f(x)$.



[Example 3] (similar to 4.1#45) Let $f(x) = (x-2)e^{(3x)}$

(A) Find the critical numbers of $f(x)$

Notice: $f(x)$ is a continuous function. So the y value $f(x)$ always exists.
The critical numbers will be the x values where $f'(c)=0$ or $f'(c)$ DNE.

Find $f'(x)$

$$f'(x) = \frac{d}{dx}\Big(\underbrace{(x-2)e^{(3x)}}_{\text{product}}\Big) \underset{\text{use Product Rule}}{=} \left(\frac{d}{dx}(x-2)\right)e^{(3x)} + (x-2)\frac{d}{dx}e^{(3x)}$$

exponential function rule #2 $\frac{d}{dx}e^{(kx)} = k\cdot e^{kx}$

$$= (1-0)e^{(3x)} + (x-2)\left(3\cdot e^{(3x)}\right)$$

$$= (1)e^{3(x)} + (3x-6)e^{(3x)}$$

$$= \left((1) + (3x-6)\right)e^{(3x)}$$

$$= (3x-5)e^{(3x)}$$

$f'(x) = (3x-5)e^{(3x)}$

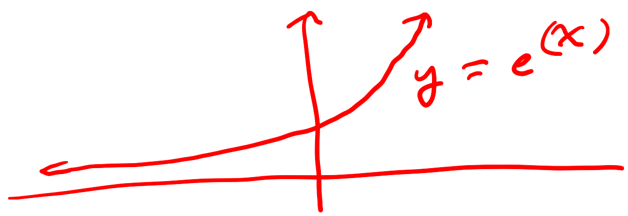
Observe that $f'(x)$ is continuous, so there are no bad x values. There are no x values that cause $f'(x)$ to not exist.

Find x values that cause $f'(x) = 0$

$f'(x) = (3x-5)e^{(3x)} = 0$

Remember the Zero product property: $a \cdot b = 0$ only when $a=0$ or $b=0$

Also remember that $e^{(3x)}$ is always positive, because $e^{(\text{anything})} > 0$



So it must be that $3x-5=0$

$3x = 5$

$x = 5/3$

So the only critical number for $f(x)$ is $x = 5/3$

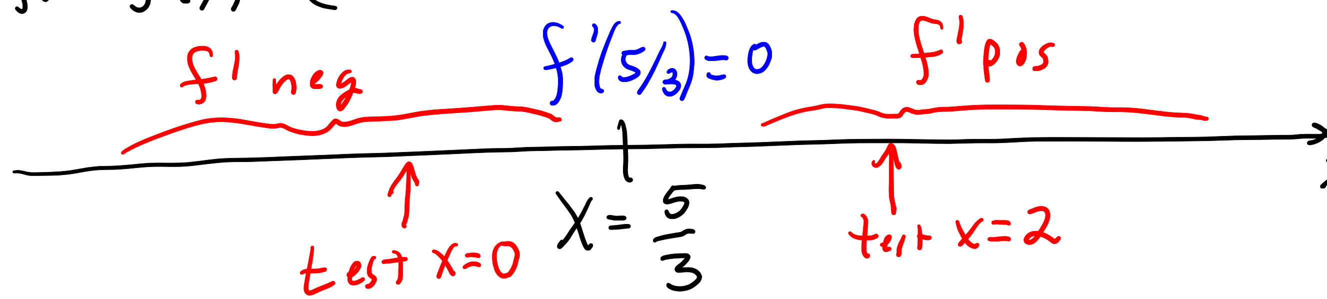
(B) Find the intervals on which $f(x)$ is increasing.

(C) Find the intervals on which $f(x)$ is decreasing.

Strategy • make sign chart for $f'(x)$

• use the sign chart for $f'(x)$ to answer the questions about $f(x)$.

Sign chart for $f'(x) = (3x-5)e^{(3x)}$



$f'(0) = (3(0)-5)e^{(3(0))} = (-5)e^{(0)} = \text{neg} \cdot \text{pos} = \text{neg}$

$f'(2) = (3(2)-5)e^{(3(2))} = (1)e^{(6)} = \text{pos} \cdot \text{pos} = \text{POS}$

$f(x)$ is decreasing on the interval $\left(-\infty, \frac{5}{3}\right)$ because f' is neg

$f(x)$ is increasing on the interval $\left(\frac{5}{3}, \infty\right)$ because f' is pos.

(D) Find the local extrema.

Local min at $x = \frac{5}{3}$ because f' changes from neg to zero to positive there.

$f(x) = (x-2)e^{(3x)}$

$f(\ ) = ((\ )-2)e^{(3(\ ))}$

empty version

The local min is the y value

$$y = f\left(\frac{5}{3}\right)$$

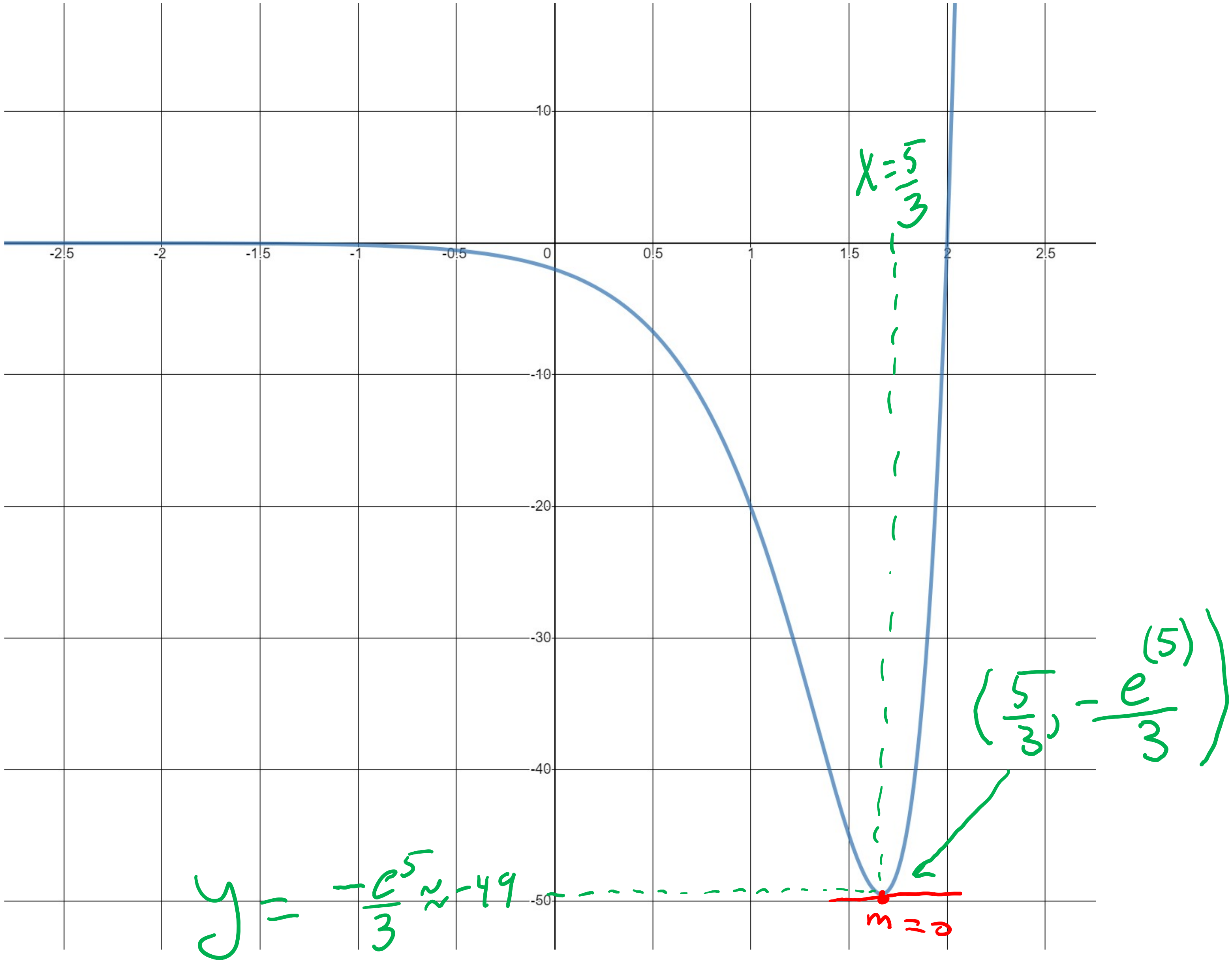
$$= \left(\left(\frac{5}{3}\right)-2\right)e^{\left(3\left(\frac{5}{3}\right)\right)}$$

$$= \left(-\frac{1}{3}\right)e^{(5)}$$

The local min is $f\left(\frac{5}{3}\right) = -\frac{e^5}{3}$

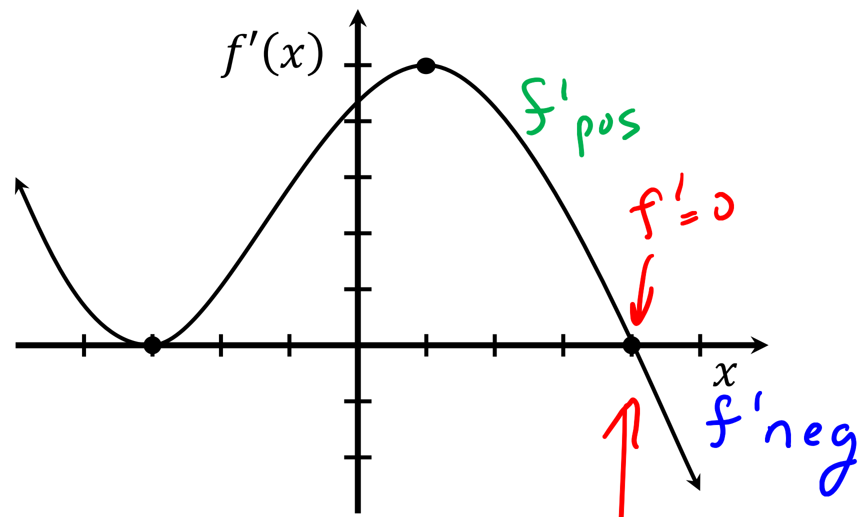
note: $-\frac{e^5}{3} \approx -49$

(E) Illustrate on the given graph of $f(x)$.



[Example 4] (similar to 4.1#77) Revisit **[Example 6]** from the Video for H53:

For the given graph of $f'(x)$



(A) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

(B) Find the x coordinates of the local extrema for $f(x)$

(C) Sketch a possible graph of $f(x)$.

this x intercept on the graph of $f'(x)$ corresponds to a local max on the graph of $f(x)$

Recall that when we discussed this example in the video for H53, it was useful to fill in this table.

x	sign of $f'(x)$ (circle one)	conclusions about behavior of function $f(x)$	
$x < -3$	pos neg zero	f increasing	
$x = -3$	pos neg zero	f has horiz tangent	not a max or a min
$-3 < x < 1$	pos neg zero	f increasing	
$x = 1$	pos neg zero		
$1 < x < 4$	pos neg zero		
$x = 4$	pos neg zero	f has horiz tangent	
$4 < x$	pos neg zero	f decreasing	

See the video for H53 to see how this table was used to answer questions **(A)** and **(C)**.

The only thing new in the current revisiting of the example is that we are asked question **(B)**.

We can answer that question by using the ***First-Derivative Test for Local Extrema.***

f(x) has local max at x=4 because
f'(x) changes from pos to zero to neg

[Example 5] (similar to 4.1#85) For the function $f(x) = 3x + \dfrac{75}{x}$

(A) Find the critical numbers of $f(x)$

Solution find $f'(x)$

First rewrite $f(x) = 3x + \dfrac{75}{x} = 3x + 75x^{-1}$

positive exponent form

power function form

Now find $f'(x) = \dfrac{d}{dx}\left(3x + 75x^{-1}\right) =$

constant multiple rule

$= 3\dfrac{d}{dx}x + 75\dfrac{d}{dx}x^{-1}$

power rule

$= 3(1) + 75\left((-1)x^{-1-1}\right)$

$= 3 - 75x^{-2}$ power function form

$= 3 - \dfrac{75}{x^2}$ positive exponent form

$$f'(x) = 3 - \frac{75}{x^2}$$

find partition numbers for $f'(x)$

Observe $f'(0) = 3 - \frac{75}{(0)^2}$ Does not exist

So $x = 0$ is a partition number for $f'(x)$

Set $f'(x) = 0$ and solve for x

$$f'(x) = 3 - \frac{75}{x^2} = 0$$

$$3 = \frac{75}{x^2}$$

$$x^2 = \frac{75}{3} = 25$$

Solutions $x = 5$, $x = -5$

The partition numbers for $f'(x)$ are $x = -5$, $x = 0$, $x = 5$

The critical numbers for $f(x)$ will be the
x values that
- are partition numbers for $f'(x)$
- are also in the <u>domain</u> of $f(x)$

$$f(x) = 3x + \frac{75}{x}$$

$$f(-5) = 3(-5) + \frac{75}{(-5)} = -15 + (-15) = -30$$

$$f(0) = 3(0) + \frac{75}{(0)} \quad \text{Does not exist}$$

$$f(5) = 3(5) + \frac{75}{5} = 15 + 15 = 30$$

So the critical numbers for $f(x)$ are $x = -5$, $x = 5$

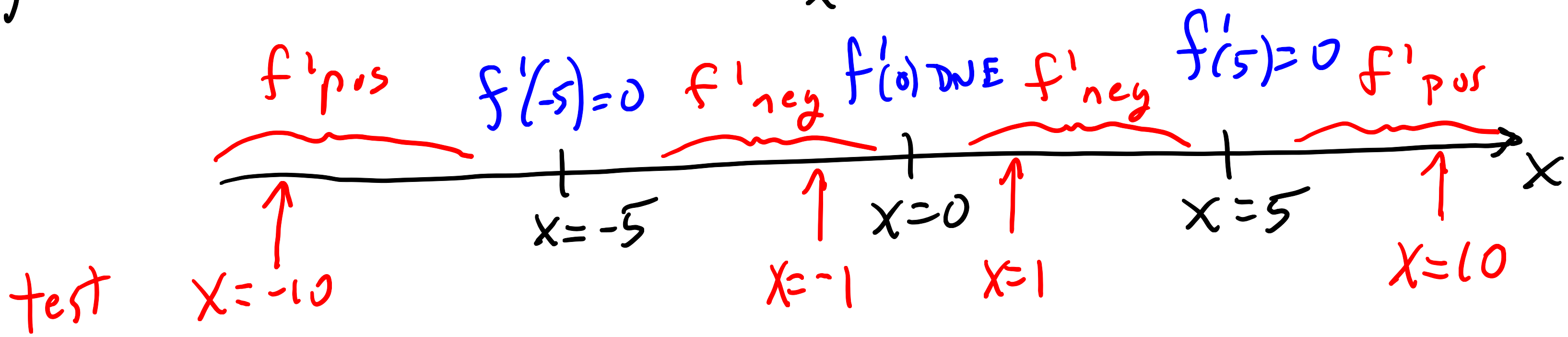
(B) Find the intervals on which $f(x)$ is increasing.

(C) Find the intervals on which $f(x)$ is decreasing.

Strategy: • make sign chart for $f'(x)$

• use that sign chart to answer the questions about $f(x)$

Sign chart for $f'(x) = 3 - \frac{75}{x^2}$



$f'(-10) = 3 - \frac{75}{(-10)^2} = 3 - \frac{75}{100} = 3 - .75 = \text{pos}$

$f'(-1) = 3 - \frac{75}{(-1)^2} = 3 - \frac{75}{1} = 3 - 75 = \text{neg}$

$f'(1) = 3 - \frac{75}{(1)^2} = 3 - 75 = \text{neg}$

$f'(10) = 3 - \frac{75}{(10)^2} = 3 - \frac{75}{100} = 3 - .75 = \text{pos}$

$f(x)$ is increasing on the intervals $(-\infty,-5)$ and $(5,\infty)$ because f' is pos.

$f(x)$ is decreasing on the intervals $(-5,0)$ and $(0,5)$ because f' is neg

(D) Find the local extrema.

The local extrema can only occur at the critical numbers for $f(x)$. They are $x=-5$, $x=5$,

Local max at $x=-5$ because f' changes pos to zero to neg

Local min at $x=5$ because f' changes neg to zero to pos.

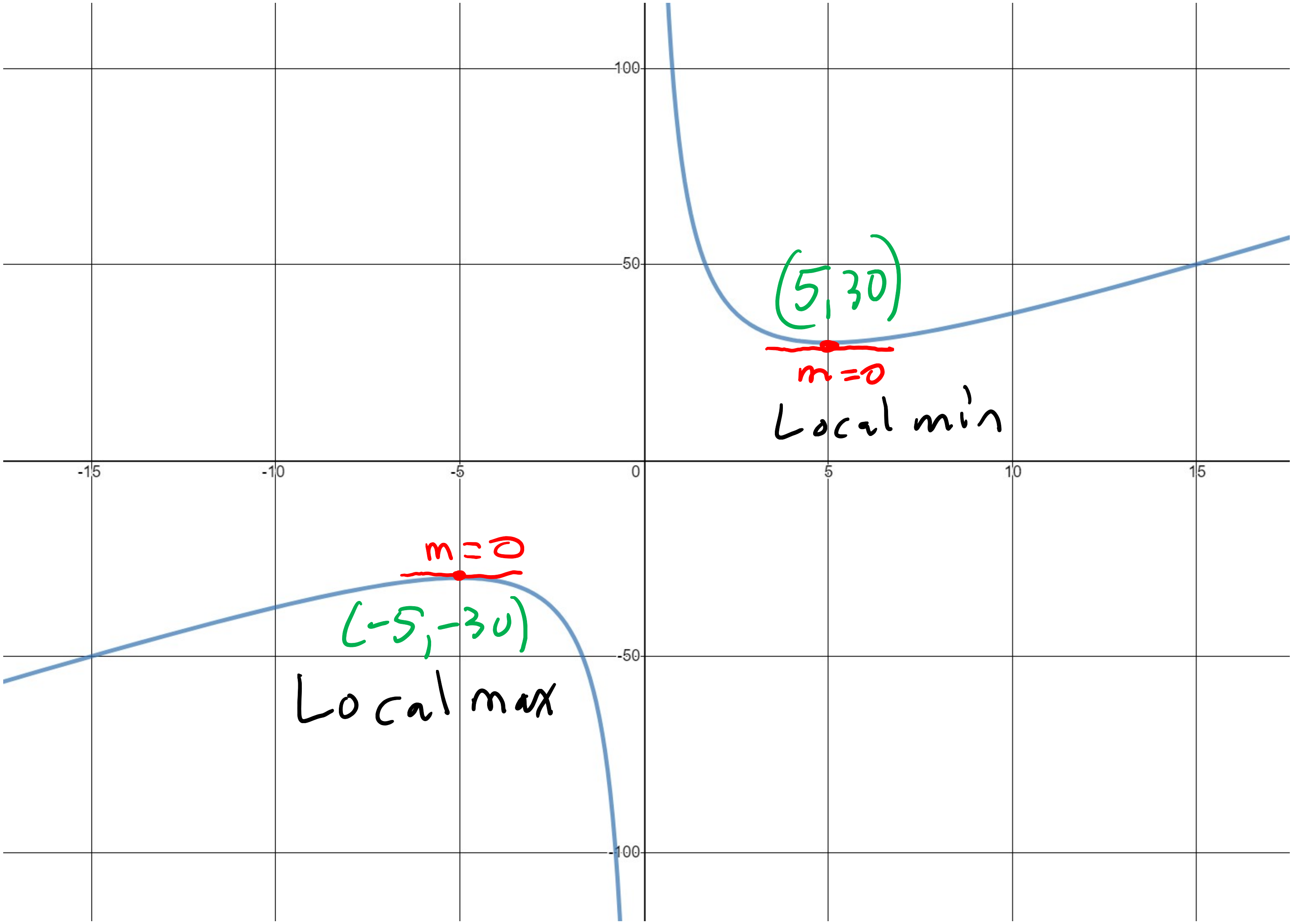
The local extrema are the y values

The local max is $f(-5)=-30$

The local min is $f(5)=30$

observe: the local max is lower than the local min!!

(E) Illustrate on the given graph of $f(x)$.



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[Example 6](Similar to 4.1#97) A drug is administered to a patient by a pill. The drug concentration in the bloodstream is described by the function

$$C(t) = \frac{0.23t}{t^2 + 9} \qquad t \geq 0$$

where t is the time in hours after the pill is taken and $C(t)$ is the drug concentration in the bloodstream (in milligrams/liter) at time t.

(A) Find critical numbers for $C(t)$

Find $C'(t) = \frac{d}{dt} \frac{0.23t}{t^2+9} = \ldots = \frac{0.23(9-t^2)}{(t^2+9)^2} = \frac{0.23(3-t)(3+t)}{(t^2+9)}$

quotient rule

Partition numbers for $C'(t)$

There are no x values that cause $C'(t)$ to not exist because the denominator will never be zero.

The only partition numbers for $C'(t)$ are $t=-3$ and $t=3$ because they cause $C'(t)=0$

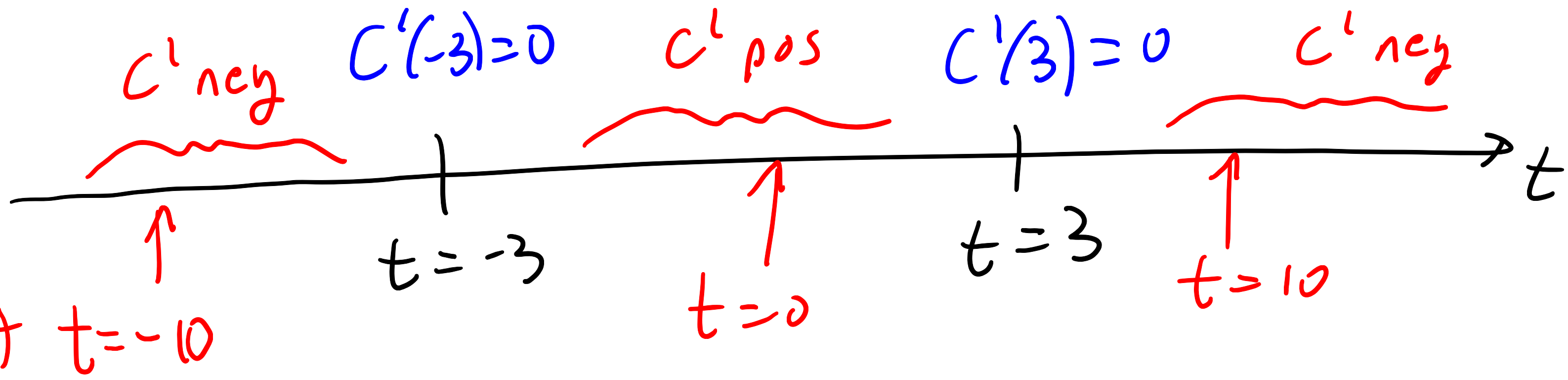
$C(-3)$ exists and $C(3)$ exists, because the denominator is not zero.

So the critical numbers for $C(t)$ are $t=-3, t=3$.

(B) Find intervals where $C(t)$ is increasing.

(C) Find intervals where $C(t)$ is decreasing.

make sign chart for $c'(t) = \dfrac{0.23(3-t)(3+t)}{(t^2+9)^2}$



$c'(-10) = \cdots = \text{neg}$

$c'(0) = \cdots = \text{pos}$

$c'(10) = \cdots = \text{neg}$

C is increasing on the interval $(-3, 3)$
because $c'(t)$ is pos

C is decreasing on the intervals $(-\infty, -3)$ and $(3, \infty)$
because $C'(t)$ is neg

(D) Find local extrema for $C(t)$

local min at $t=-3$ (we don't care, because we only want $t \geq 0$)

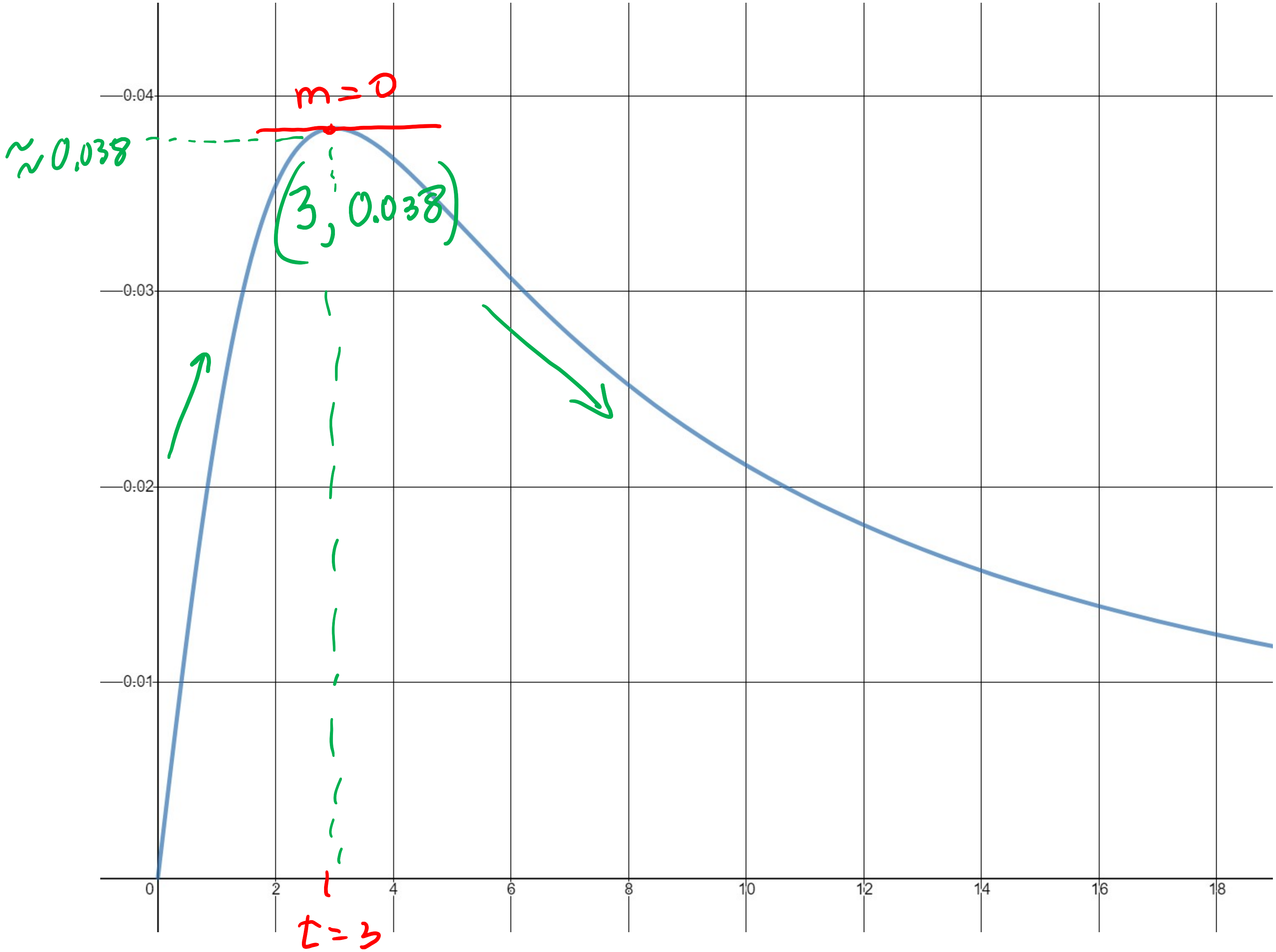
local max at $\underline{t=3}$,

The local max is the y value

$$C(3) = \frac{.23(3)}{3^2+9} = \frac{.69}{9+9} = \frac{.69}{18} \approx 0.038$$

$$C(t) = \frac{.23t}{t^2+9}$$

(E) Illustrate the results using the given graph of $C(t)$



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