Subject for this video:

Introduction to Concavity

Reading:

- **General:** Section 4.2 Second Derivatives and Graphs
- **More Specifically:** The concepts in this video are scattered throughout Section 4.2 of the book. There are no book examples similar to the examples in this video.

Homework:

H57: Introduction to Concavity (4.2#9,13,14,15,16)

Useful Section 4.1 concept discussed in previous videos

Correspondence between

sign behavior of $f'(x)$ on an interval (a, b) and behavior of graph of $f(x)$ on the interval (a, b)

- If $f'(x)$ is positive on an interval (a, b) then $f(x)$ is increasing on the interval (a, b).

- If $f'(x)$ is negative on an interval (a, b) then $f(x)$ is decreasing on the interval (a, b).

- If $f'(x)$ is zero on an interval (a, b) then $f(x)$ is constant on the interval (a, b).

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Introduction to Concavity, and Examples Involving Graphs

Definition of Concavity and Inflection Point

Words: f *is* ***concave up*** *on the interval* (a, b).

Graphical Definition: For every $x = c$, with $a < c < b$, the graph of f has a tangent line at $x = c$ and the graph of f stays above that tangent line for x-values in the interval (a, b).

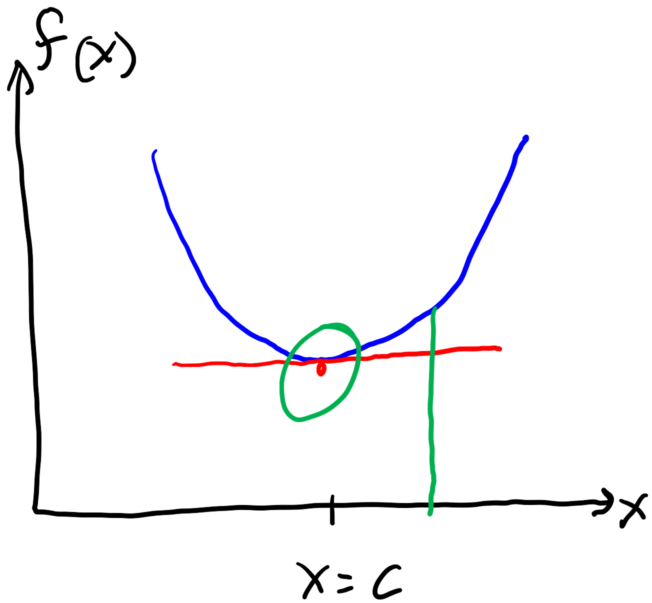
Abstract Definition: $f'(x)$ is *increasing* on the interval (a, b)

Words: f *is* ***concave down*** *on the interval* (a, b).

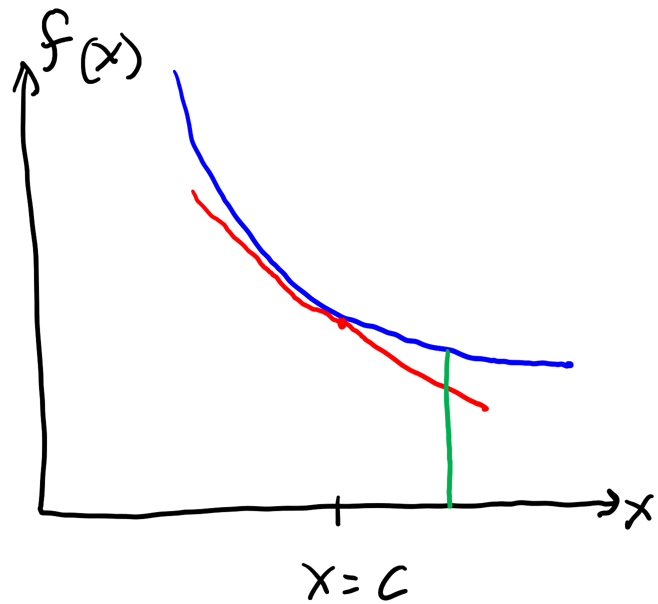
Graphical Definition: For every $x = c$, with $a < c < b$, the graph of f has a tangent line at $x = c$ and the graph of f stays below that tangent line for x-values in the interval (a, b).

Abstract Definition: $f'(x)$ is *increasing* on the interval (a, b)

Related terminology: An ***inflection point*** is point on the graph of a function where the function is continuous and the concavity changes (from up to down or from down to up.)



f(x)
x
x = c



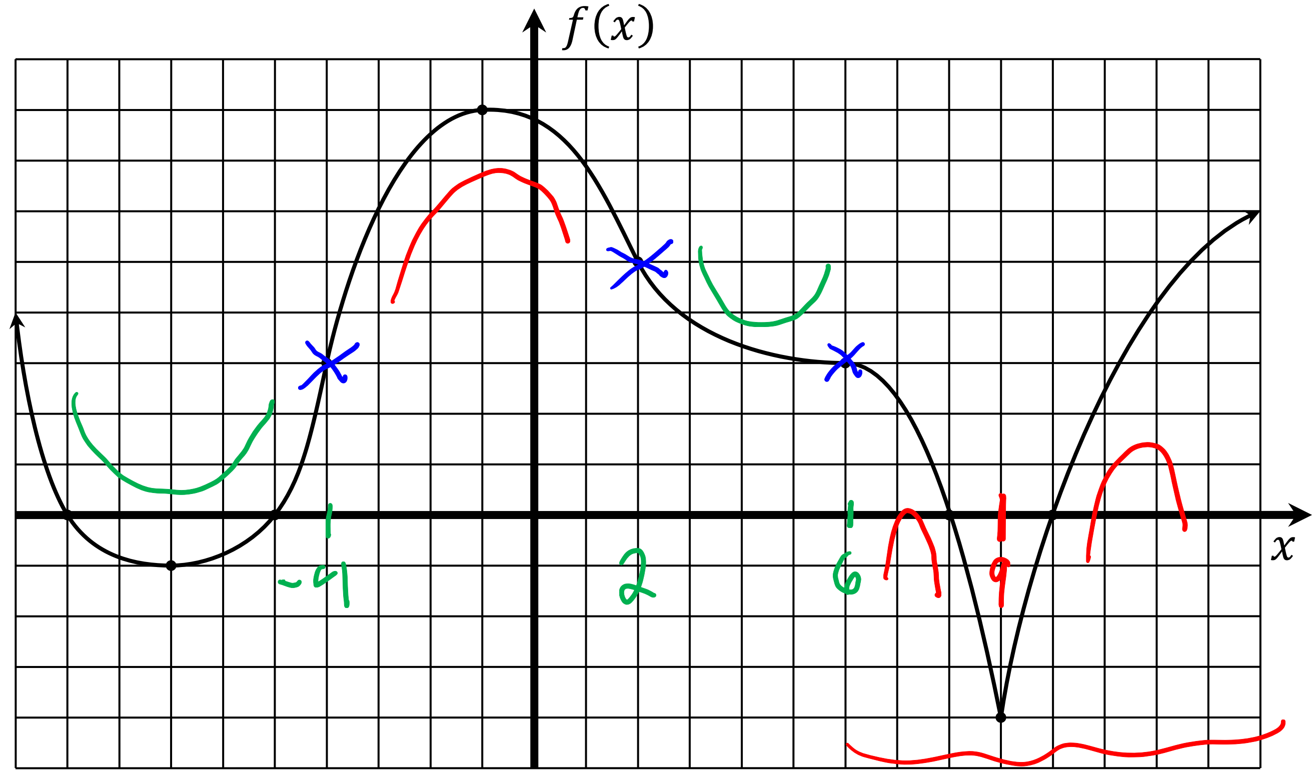
f(x)
x
x = c

Resume [Example 1] from

Video 20 and Video 53

(similar to 4.1#11 and

4.2#9,13,14,15,16)



(l) On which intervals is $f(x)$ concave up? $(-\infty,-4)$ and $(2,6)$

(m) On which intervals is $f(x)$ concave down? $(-4,2)$ and $(6,9)$ and $(9,\infty)$

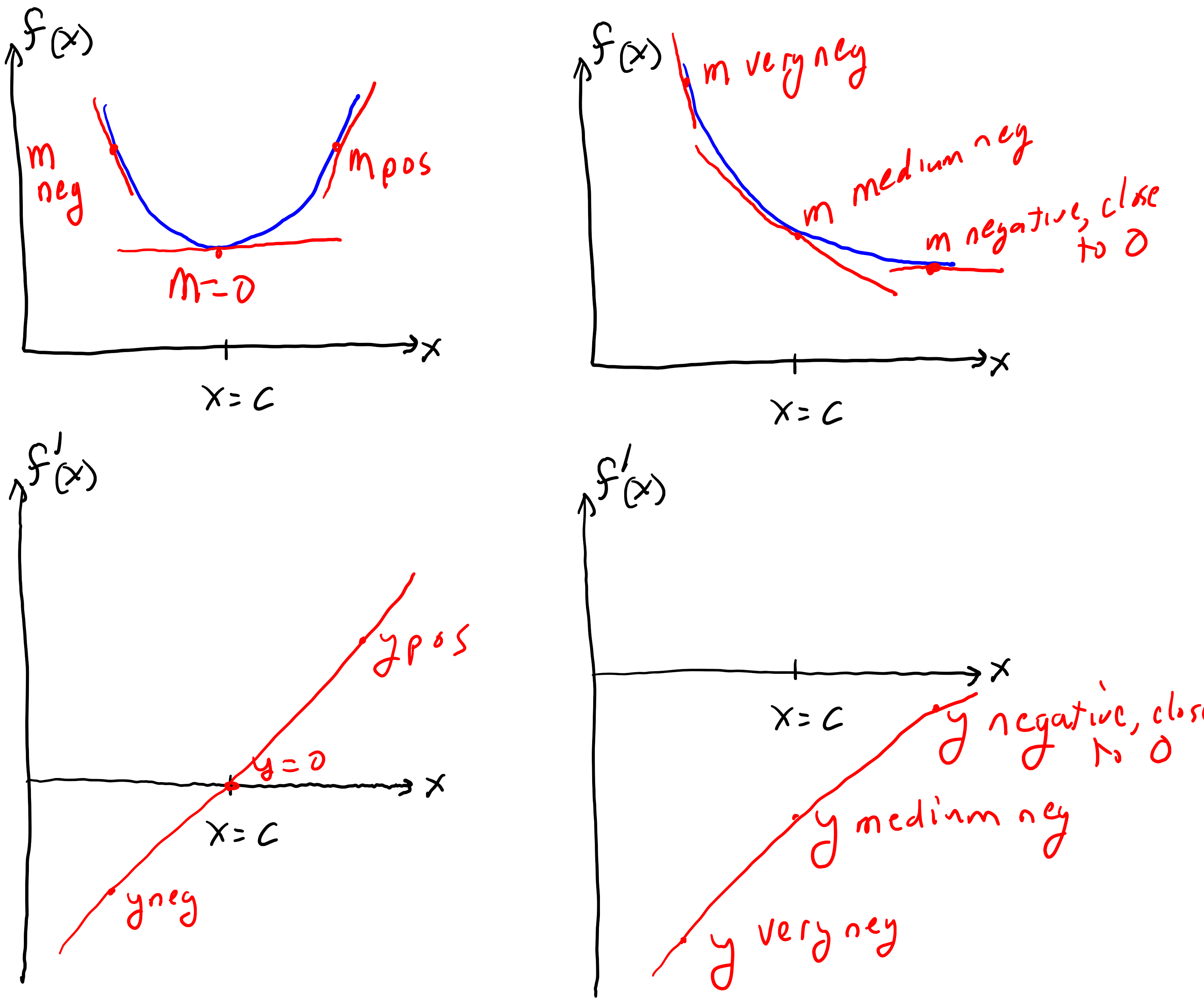
~~$(6,\infty)$~~

(n) Find the x coordinates of all inflection points on the graph of $f(x)$.

$X = -4, 2, 6$

(o) Find all inflection points on the graph of $f(x)$.

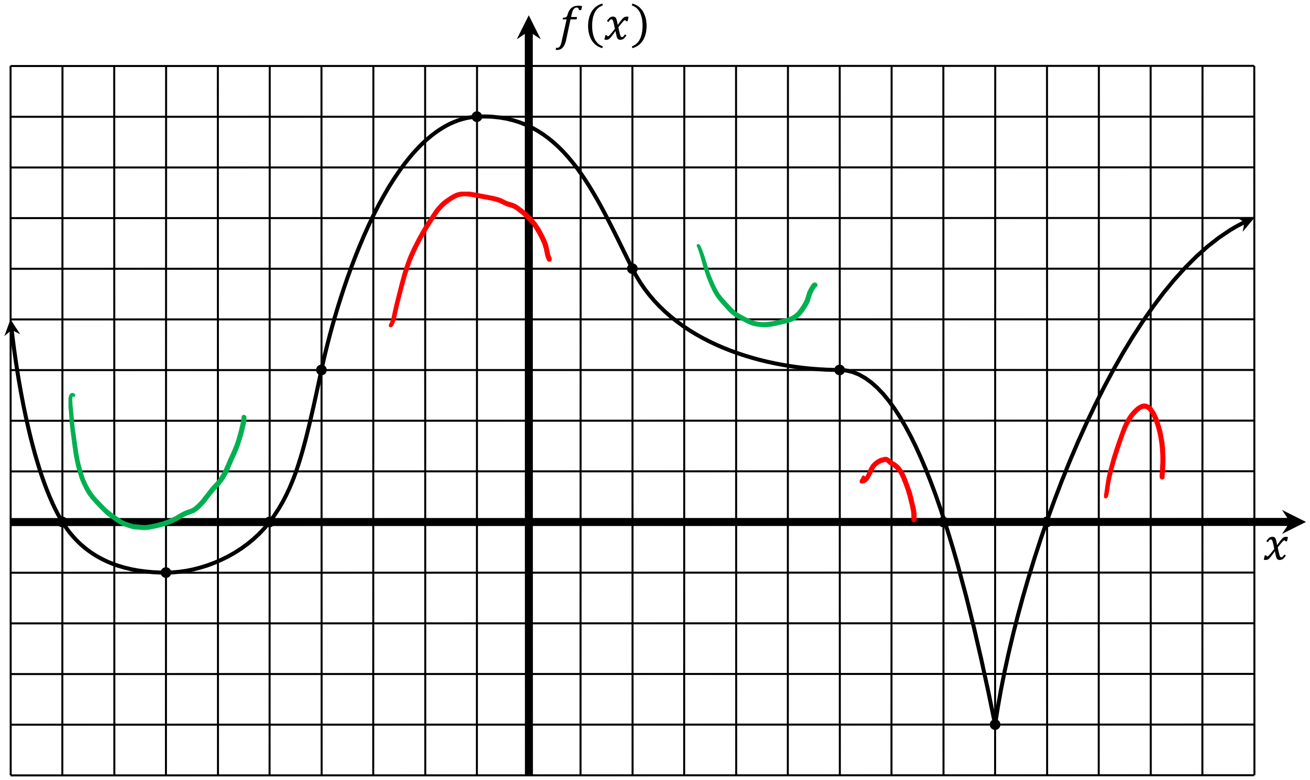
$(-4,3)$ and $(2,5)$ and $(6,3)$



f(x)
m neg
m pos
m = 0
x
x = c
f(x)
m very neg
m medium neg
m negative, close to 0
x
x = c
f'(x)
y pos
y = 0
x
x = c
y neg
f'(x)
x
x = c
y negative, close to 0
y medium neg
y very neg

In Section 4.1, we sometimes examined a given graph of a function $f(x)$, scrutinizing its *increasing & decreasing* behavior, to determine the *sign* behavior of $f'(x)$.

Now we realize that, given a graph of a function $f(x)$, we can scrutinize its *concavity* behavior to determine the *increasing & decreasing* behavior of $f'(x)$.



(p) On which intervals is $f'(x)$ increasing?

f' is increasing on $(-\infty,-4)$ and $(2,6)$ because $f(x)$ is concave up there.

(q) On which intervals is $f'(x)$ decreasing?

f' is decreasing on $(-4,2)$ and $(6,9)$ and $(9,\infty)$ because $f(x)$ is concave down there.

What if a function $f(x)$ is given by a formula, and not by a graph. Is there some way to scrutinize the *formula* for $f(x)$ and determine the *concavity*?

It turns out that there is a way.

The key is to note two things:

(1) The abstract definition of the *concavity* of $f(x)$ is in terms of the *increasing & decreasing behavior* of $f'(x)$,

(2) The *increasing & decreasing behavior* of $f'(x)$ will be related to the sign behavior of *the derivative of* $f'(x)$.

Thus, we are led to study the *derivative of* $f'(x)$.

Definition of the Second Derivcative

Words: *the second derivative of* $f(x)$.

Symbols: $f''(x), \qquad \dfrac{d^2}{dx^2} f(x)$

Meaning: $f''(x) = \dfrac{d}{dx} f'(x) = \dfrac{d}{dx}\dfrac{d}{dx} f(x)$

We immediately get the following correspondence

Correspondence between *sign behavior* of $\boldsymbol{f''(x)}$ and *concavity behavior* of $\boldsymbol{f(x)}$

- If $f''(x)$ is *positive* on an interval (a, b), then $f'(x)$ is *increasing* on the interval (a, b), which in turn means that $f(x)$ is *concave up* on the interval (a, b).
- If $f''(x)$ is *negative* on an interval (a, b), then $f'(x)$ is *decreasing* on the interval (a, b), which in turn means that $f(x)$ is *concave down* on the interval (a, b).

Subtlety in the correspondence between *sign* of $f''(x)$ and *concavity* of $f(x)$

The correspondence just presented seems simple enough, but there is some subtlet. Namely,

- If $f''(x)$ is positive on a whole interval (a, b), then it is guaranteed that $f(x)$ will be concave up on the whole interval (a, b).
- But if $f(x)$ is concave up on the whole interval (a, b), it is not guaranteed that $f''(x)$ will be positive on the whole interval (a, b).

We have run into this kind of subtlety before, back in Section 4.1

- If $f'(x)$ is positive on a whole interval (a, b), then it is guaranteed that $f(x)$ will be increasing on the whole interval (a, b).
- But if $f(x)$ is increasing on the whole interval (a, b), it is not guaranteed that $f'(x)$ will be positive on the whole interval (a, b).

It is worthwhile to consider two examples that illustrate this subtlety.

Let $f(x) = x^2$

Then observe that $f'(x) = \frac{d}{dx} x^2 = 2x$

and $f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} 2x = 2$

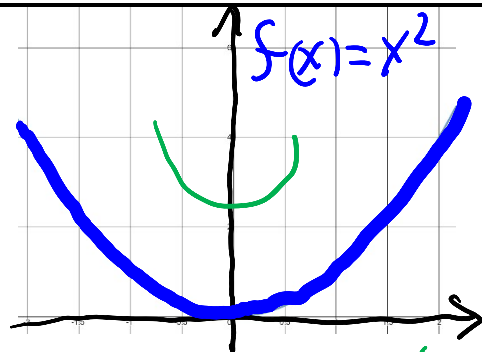
Let $g(x) = x^4$

Then observe that $g'(x) = \frac{d}{dx} x^4 = 4x^3$

and $g''(x) = \frac{d}{dx} g'(x) = \frac{d}{dx} 4x^3 = 12x^2$

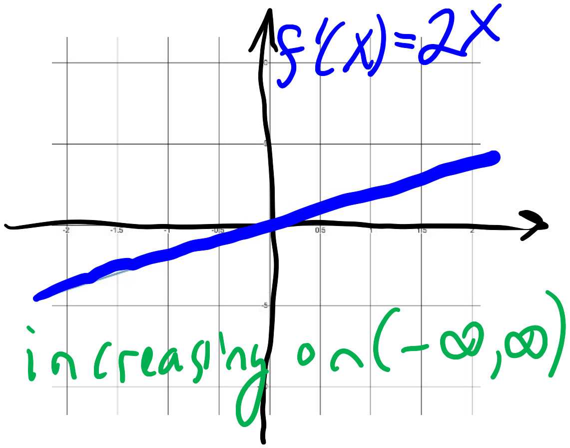
Now consider the graphs of f, f', f'' and g, g', g'' on the next page

$f(x)=x^2$



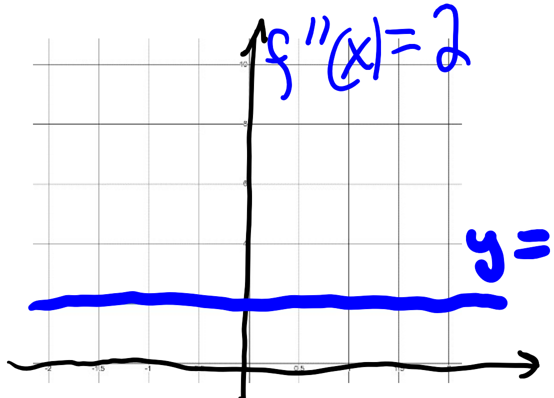
f concave up on $(-\infty,\infty)$

$f'(x)=2x$



f' increasing on $(-\infty,\infty)$

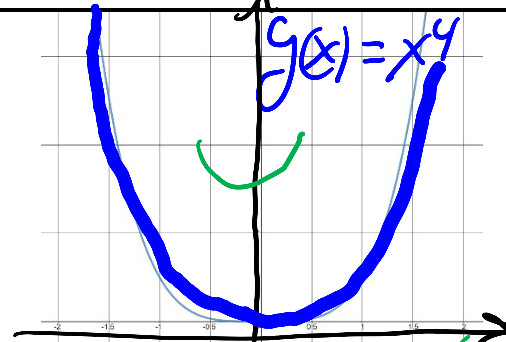
$f''(x)=2$



$y=2$

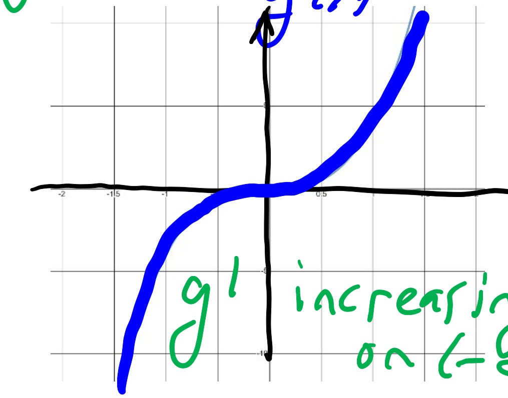
f'' positive on $(-\infty,\infty)$

$g(x)=x^4$



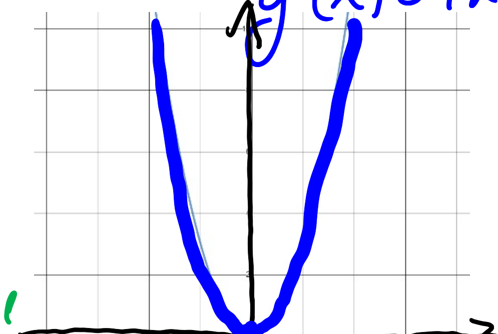
g concave up on $(-\infty,\infty)$

$g'(x)=4x^3$



g' increasing on $(-\infty,\infty)$

$g''(x)=12x^2$



g'' positive on $(-\infty,0)$ 0 $(0,\infty)$

Observe that

$f(x)$ is concave up on the whole interval $(-\infty, \infty)$

$f'(x)$ is increasing on the whole interval $(-\infty, \infty)$

$f''(x)$ is positive on the whole interval $(-\infty, \infty)$

$g(x)$ is concave up on the whole interval $(-\infty, \infty)$

$g'(x)$ is increasing on the whole interval $(-\infty, \infty)$

$g''(x)$ is positive on the intervals $(-\infty, 0)$ and $(0, \infty)$, but not on the whole interval $(-\infty, \infty)$

The conclusion from this is that appearances can be deceiving.

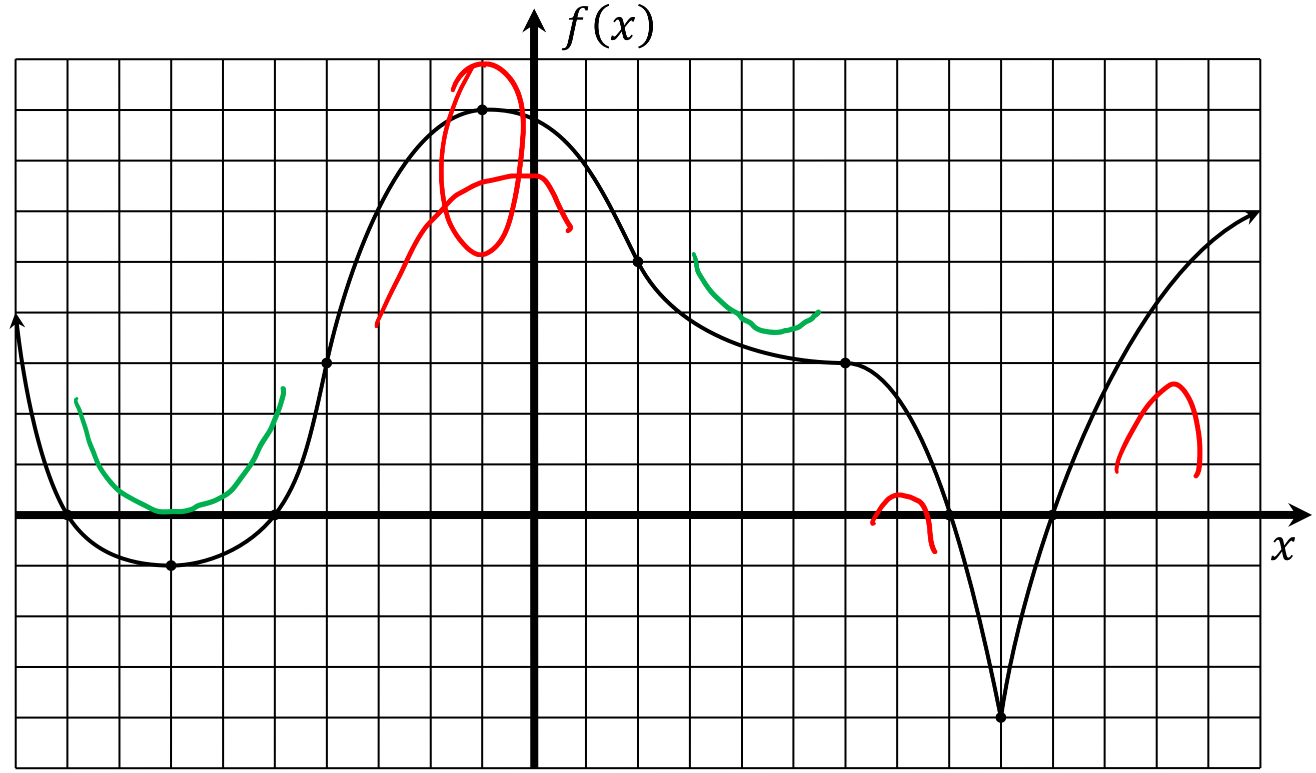
For that reason, one has to be a little cautious trying to draw conclusions about the sign behavior of $f''(x)$ from a given graph of $f(x)$. That caution shows up in the rather peculiar wording of some of the questions in your homework problem 4.2#9. The following example has the same kind of wording.

Resume [Example 1] from

Video 20 and Video 53

(similar to 4.1#11 and

4.2#9,13,14,15,16)



(r) On which intervals is $f''(x) > 0$? (Assume that $f''(-7) > 0$.)

$f''(x) > 0$ on $(-\infty, -4)$ and $(2, 6)$, because f is concave up there.

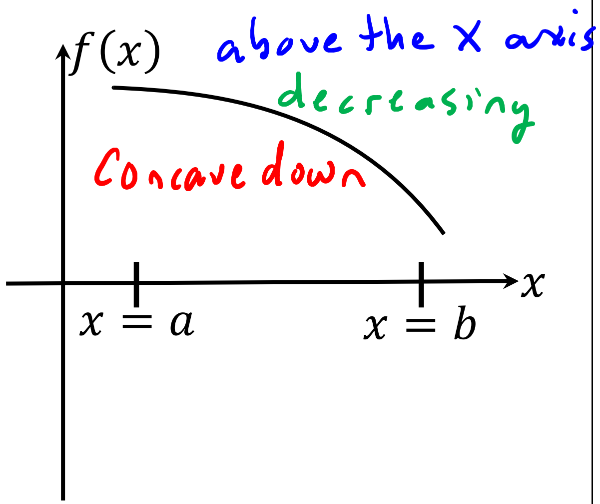
(s) On which intervals is $f''(x) < 0$? (Assume that $f''(-1) < 0$.)

$f''(x) < 0$ on $(-4, 2)$ and $(6, 9)$ and $(9, \infty)$ because f is concave down there.

[Example 2] (Similar to 4.2#13-16) Given graph of $f(x)$, tell the sign of f, f', f''.

For the given graphs, circle the correct word *positive* or *negative* in each of the three sentences.

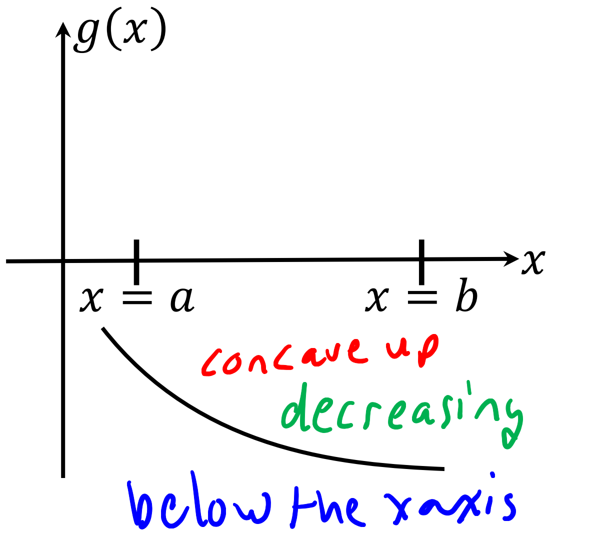
Cross out the incorrect word *positive* or *negative* in each sentence.



$f(x)$ is *positive* ~~*negative*~~ on the interval (a, b).

$f'(x)$ is ~~*positive*~~ *negative* on the interval (a, b).

$f''(x)$ is ~~*positive*~~ *negative* on the interval (a, b).



$g(x)$ is ~~*positive*~~ *negative* on the interval (a, b).

$g'(x)$ is ~~*positive*~~ *negative* on the interval (a, b).

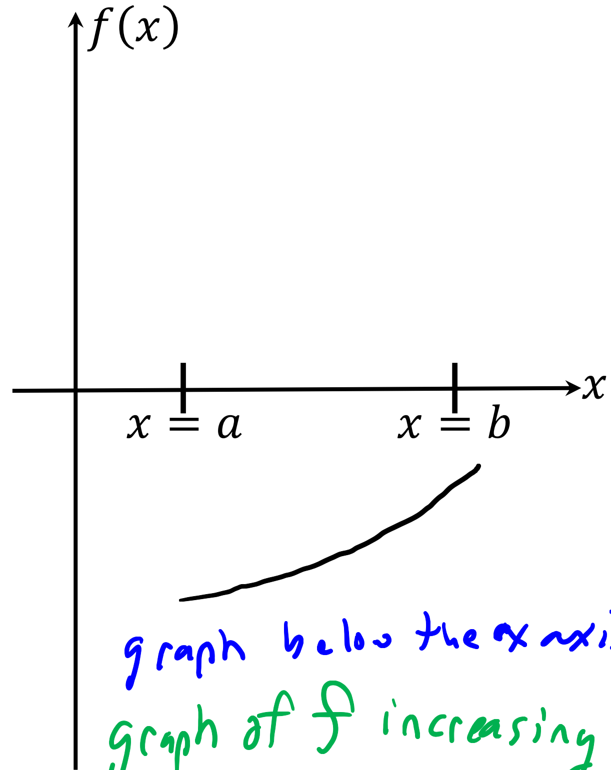
$g''(x)$ is *positive* ~~*negative*~~ on the interval (a, b).

[Example 3] (Similar to 4.2#13-16) Given sign of f, f', f'', sketch possible graph of $f(x)$.

Given the following information about the behavior of the function $f(x)$ on the interval (a, b):

$f(x) < 0$ and $f'(x) > 0$ and $f''(x) < 0$ on the interval (a, b)

Sketch a possible graph of $f(x)$ on the axes provided.



graph below the x axis

graph of f increasing

f concave up

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