**Subject for this video:**

**Given formula for $f(x)$, determine concavity and find inflection points**

**Reading:**

- **General:** Section 4.2 Second Derivatives and Graphs
- **More Specifically:** Examples 1,2,3.

**Homework:**

H59: Given formula for $f(x)$, determine concavity and find inflection points (4.2#33,35,37)

**[Example 1]** (similar to 4.1#49,51,53,55 and 4.2#33,35)

Revisit the function that was used for **[Example 2]** in the Video for H54 and the Video for H56.

$$f(x) = -x^4 + 4x^3$$

**(A)** Find the local extrema of $f(x)$.

Strategy:
- find $f'(x)$
- make sign chart for $f'(x)$
- Use sign chart for $f'(x)$ to find x coordinates of extrema in $f(x)$
- Use formula for $f(x)$ to find y coordinates of extrema in $f(x)$.

$$f'(x) = \frac{d}{dx}\left(-x^4 + 4x^3\right) = -4x^3 + 12x^2 = -4x^2(x-3)$$

Partition numbers for $f'(x)$ are $x = 0$, $x = 3$

Sign chart for $f'(x) = -4x^2(x-3)$

$f'$ pos $\quad f'(0)=0 \quad f'$ pos $\quad f'(3)=0 \quad f'$ neg

$x=0 \qquad x=3 \qquad x$

test $x=-1 \qquad x=1 \qquad x=4$

$$f'(-1) = -4(-1)^2((-1)-3) = -4(1)(-4) = \text{pos}$$

$$f'(1) = -4(1)^2((1)-3) = -4(1)(-2) = \text{pos}$$

$$f'(4) = -4(4)^2((4)-3) = -4(16)(1) = \text{neg}$$

$f(x)$ will have a local max at $x=3$ because $f'(x)$ changes from pos to zero to neg

Use $f(x)$ formula to get $y$ coordinates

$$f(x) = -x^4 + 4x^3$$

$$f(\quad) = -(\quad) + 4(\quad)^3$$ empty version

$$f(3) = -(3)^4 + 4(3)^3$$
$$= -81 + 4(27)$$
$$= -81 + 108$$
$$= 27$$

So the local max is at $(x,y) = (3, 27)$

That is the local max is $f(3) = 27$.

**(B)** Find the intervals on which the graph of $f(x)$ is concave upward.

**(C)** Find the intervals on which the graph of $f(x)$ is concave downward.

**(D)** Find the $x$ coordinates of all inflection points in the graph of $f(x)$.

Strategy
- find $f''(x)$
- make sign chart for $f''(x)$
- use that sign chart to answer B, C, D

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(-4x^3 + 12x^2\right) = -12x^2 + 24x$$

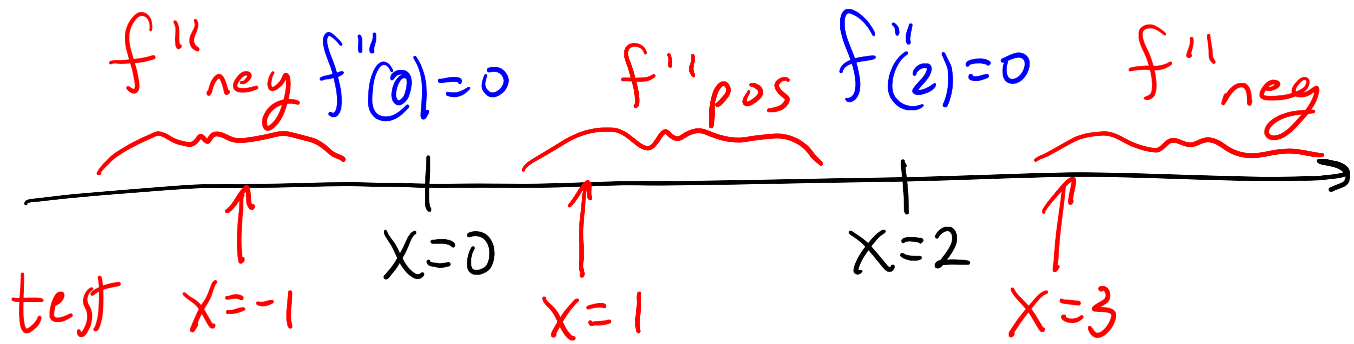
$$= -12x(x-2)$$

Partition numbers for $f''(x)$: Set $f''(x) = 0$ and solve for $x$.

$$0 = -12x(x-2)$$

Solutions: $x = 0$, $x = 2$

Sign chart for $\underline{f''(x) = -12x^2 + 24x = -12x(x-2)}$




$$f''(-1) = -12(-1)((-1)-2) = (-12)(-1)(-3) = \text{neg}$$

$$f''(1) = -12(1)((1)-2) = (-12)(1)(-1) = \text{pos}$$

$$f''(3) = -12(3)((3)-2) = -12(3)(1) = \text{neg}$$

$f(x)$ is concave up on interval $(0,2)$ because $f''$ is positive

$f(x)$ is concave down on intervals $(-\infty, 0)$ and $(0, \infty)$ because $f''(x)$ is negative there.

$f(x)$ has inflection points at $x=0$, $x=2$ because $f''(x)$ changes sign there and $f(x)$ exists.

**(E)** Find all inflection points in the graph of $f(x)$. (the $(x,y)$ coordinates.)

$$f(x) = -x^4 + 4x^3$$

$$f(\ ) = -(\ )^4 + 4(\ )^3$$ empty version

$$f(0) = -(0)^4 + 4(0)^3 = 0$$

$$f(2) = -(2)^4 + 4(2)^3 = -16 + 4(8) =$$

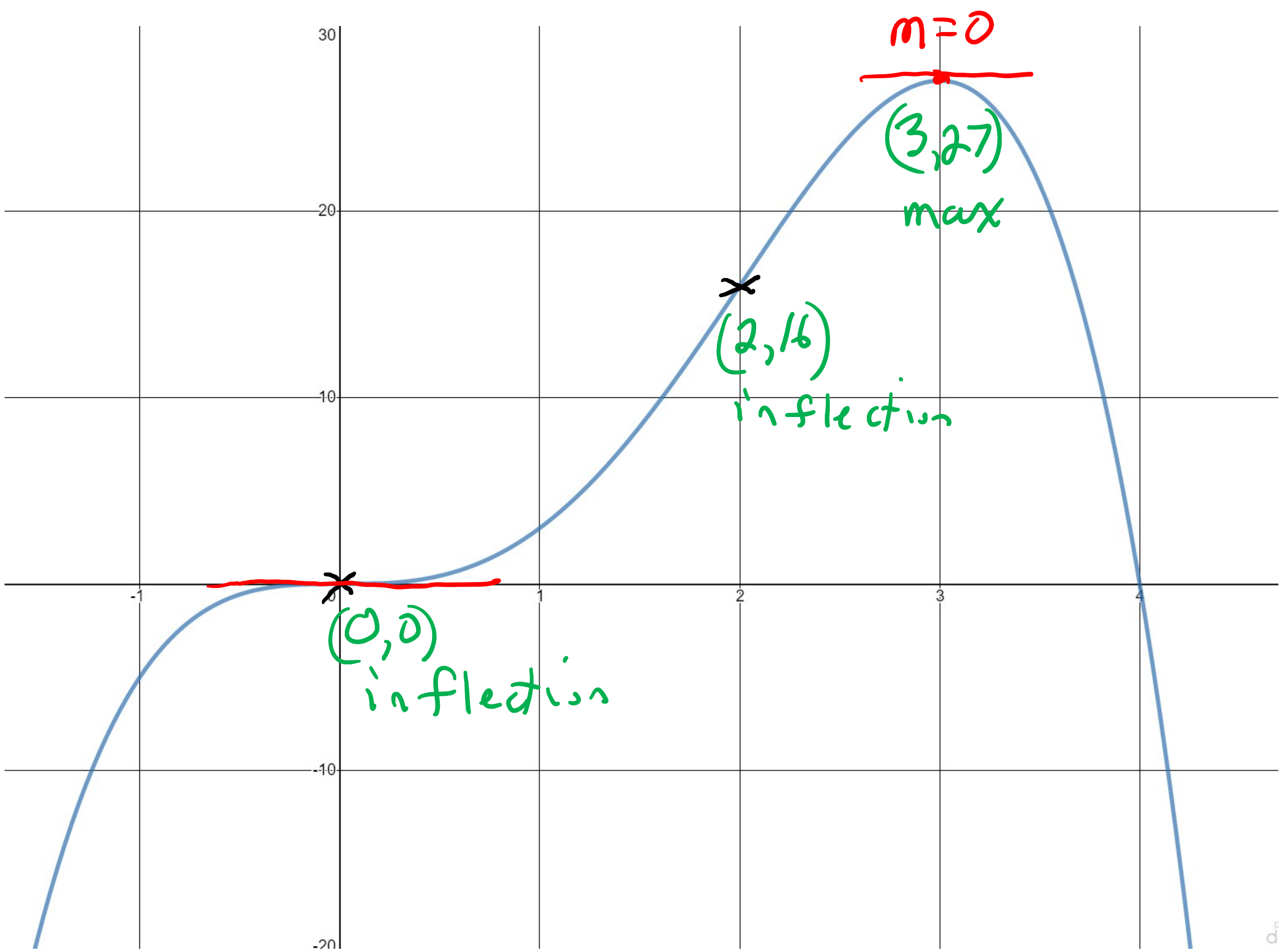
$$= -16 + 32$$

$$= 16$$

So the inflection points are at

$(x,y) = (0,0)$ and $(x,y) = (2,16)$

**(F)** Illustrate all your results on the given graph of $f(x)$.



**[Example 2]** (similar 4.2#37) Let $f(x) = \ln(x^2 + 6x + 13)$

**(A)** Find the local extrema of $f(x)$.

Strategy:
- find $f'(x)$
- make sign chart for $f'(x)$
- Use sign chart for $f'(x)$ to find x coordinates of extrema in $f(x)$
- Use formula for $f(x)$ to find y coordinates of extrema in $f(x)$.

$$f'(x) = \frac{d}{dx}\ln(x^2+6x+13)$$

$$= \frac{d}{dx}\,\text{outer}(\text{inner}(x))$$

chainrule

$$= \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)$$

$$= \frac{1}{(x^2+6x+13)} \cdot (2x+6)$$

$$= \frac{2x+6}{x^2+6x+13}$$

Chain Rule details

$\text{inner}(x) = x^2+6x+13$

$\text{inner}'(x) = 2x+6$

$\text{outer}(\;\;) = \ln(\;\;\;)$

$\text{outer}'(\;\;) = \dfrac{1}{(\;\;)}$

empty version

Need to find partition numbers for $f'(x) = \dfrac{2x+6}{x^2+6x+13}$

Are there any x coordinates that cause $f'(x)$ to not exist?

Observe that there are <u>no</u> x values that cause the denominator to be zero.

The equation $x^2+6x+13=0$ has no solutions.

(why? Can try to factor it, or use quadratic formula)

So there are no x values that cause $f'(x)$ to not exist.

The only partition numbers for $f'(x)$ will be the numbers that cause the <u>numerator</u> to be zero

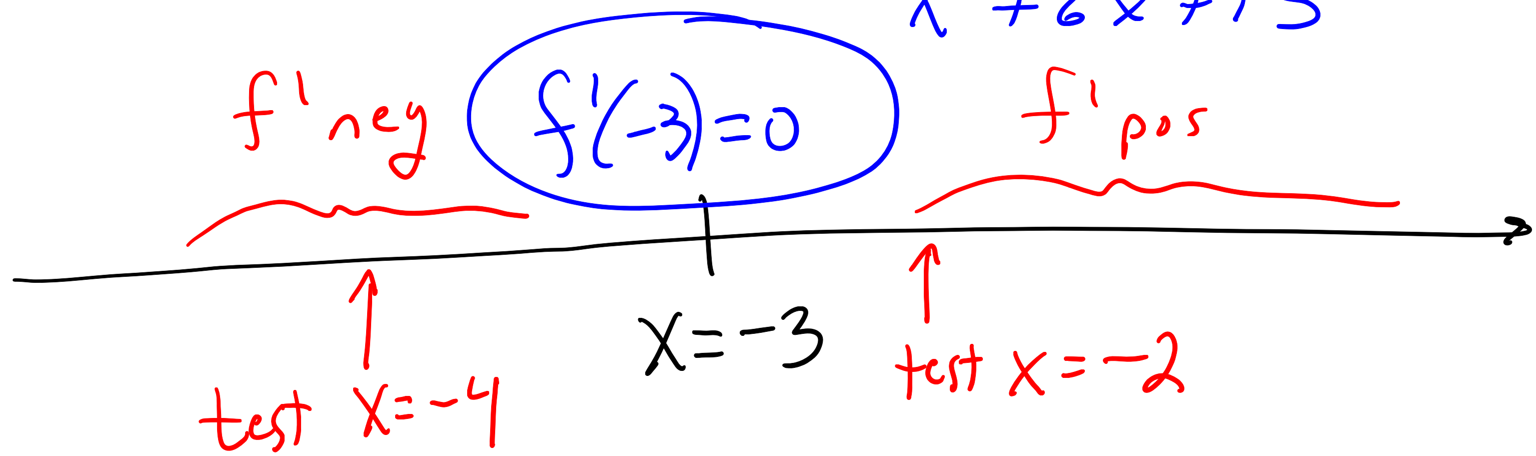
$$2x+6=0$$

$$x+3=0$$

$$x=-3$$

So the only partition number for $f'(x)$ is $x=-3$

Sign chart for $f'(x) = \dfrac{2x+6}{x^2+6x+13}$




$$f'(-4) = \frac{2(-4)+6}{(-4)^2+6(-4)+13} = \frac{-8+6}{16-24+13} = \frac{-2}{5} = \text{neg}$$

$$f'(-2) = \frac{2(-2)+6}{(-2)^2+6(-2)+13} = \frac{-4+6}{4-12+13} = \frac{2}{5} = \text{pos}$$

Observe: At $x=-3$

$f'$ changes from neg to zero to pos

$f(-3)$ exists because $f'(-3)=0$

So there is a local min at $x=-3$

Find the y coordinate of the local min using the formula $f(x) = \ln(x^2+6x+13)$

$$f(-3) = \ln((-3)^2 + 6(-3) + 13)$$

$$= \ln(9 - 18 + 13)$$

$$= \ln(4)$$

So the local min is at $(x,y) = (-3, \ln(4))$

The local min is $f(-3) = \ln(4)$

**(B)** Find the intervals on which the graph of $f(x)$ is concave upward.

**(C)** Find the intervals on which the graph of $f(x)$ is concave downward.

**(D)** Find the $x$ coordinates of all inflection points in the graph of $f(x)$.

Strategy: find $f''(x)$

make sign chart for $f''(x)$

use sign chart to answer (B), (C), (D).

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \frac{2x+6}{x^2+6x+13} =$$

quotient rule

$$= \frac{\left(\frac{d}{dx} 2x+6\right)(x^2+6x+13) - (2x+6)\left(\frac{d}{dx} x^2+6x+13\right)}{(x^2+6x+13)^2}$$

$$= \frac{(2)(x^2+6x+13) - (2x+6)(2x+6)}{(x^2+6x+13)^2} =$$

$$= \frac{2x^2+12x+26 - (4x^2+12x+12x+36)}{(x^2+6x+13)^2}$$

$$= \frac{-2x^2-12x-10}{(x^2+6x+13)^2}$$

$$= \frac{-2(x^2+6x+5)}{(x^2+6x+13)^2}$$

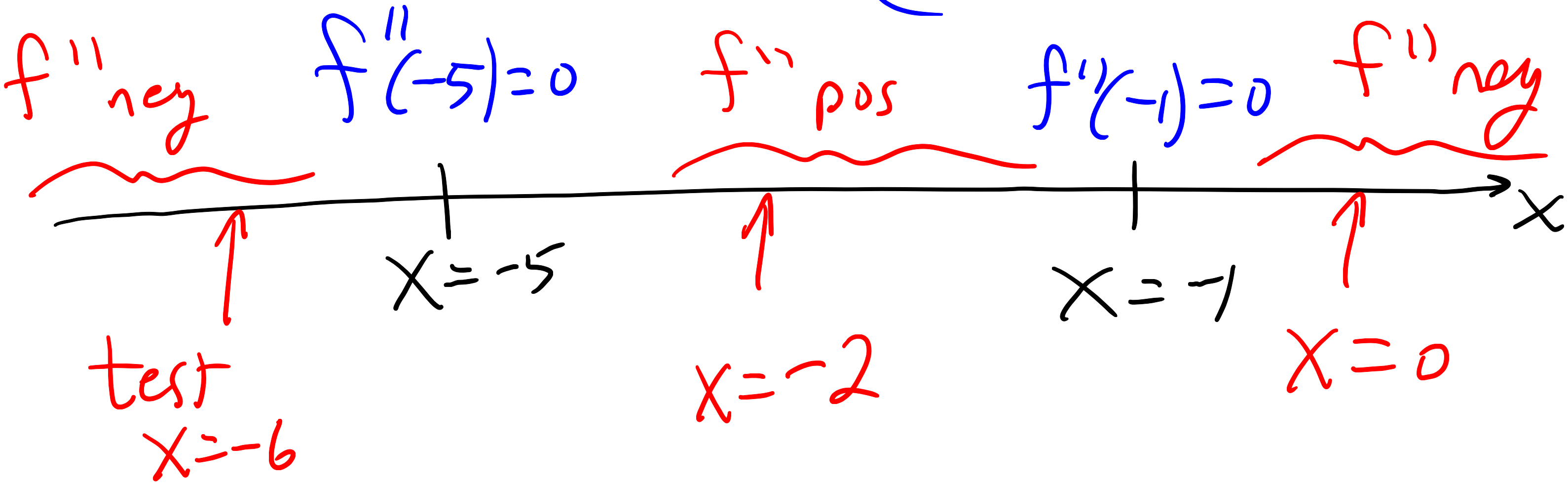
$$f''(x) = \frac{-2(x+1)(x+5)}{(x^2+6x+13)^2}$$

Partition numbers for $f''(x)$:

- No $x$ values cause $f''(x)$ to not exist because denominator is never zero
- we see that $f''(-1)=0$ and $f''(-5)=0$ because those $x$ values cause the <u>numerator</u> to be 0.

So partition numbers for $f''(x)$ are $x=-1$, $x=-5$.

Sign chart for $f''(x) = \dfrac{-2(x+1)(x+5)}{(x^2+6x+13)^2}$




$$f''(-6) = \frac{-2((-6)+1)((-6)+5)}{((-6)^2+6(-6)+13)^2} = \frac{-2(-5)(-1)}{\text{pos}} = \text{neg}$$

$$f''(-2) = \frac{-2((-2)+1)((-2)+5)}{((-2)^2+6(-2)+13)^2} = \frac{-2(-1)(3)}{\text{pos}} = \text{pos}$$

$$f''(0) = \frac{-2((0)+1)((0)+5)}{((0)^2+6(0)+13)^2} = \frac{-2(1)(5)}{\text{pos}} = \text{neg}$$

Conclude that

$f(x)$ is concave down on $(-\infty,-5)$ and $(-1,\infty)$ because $f''$ is neg

$f(x)$ is concave up on $(-5,-1)$ because $f''$ is pos

$f$ has inflection points at $x=-5$ and $x=-1$
because $f''(x)$ changes sign and
$f(x)$ exists (because $f''(x)$ exists)

**(E)** Find all inflection points in the graph of $f(x)$.

Need to use $f(x) = \ln(x^2+6x+13)$ to get $y$ coordinates

$$f(-5) = \ln((-5)^2 + 6(-5) + 13)$$
$$= \ln(25 - 30 + 13) = \ln(8)$$

$$f(-1) = \ln((-1)^2 + 6(-1) + 13)$$
$$= \ln(1 - 6 + 13)$$
$$= \ln(8)$$

So the inflection points are at

$(x, y) = (-5, \ln(8))$ and $(x, y) = (-1, \ln(8))$

**(F)** Illustrate all your results on the given graph of $f(x)$.