Subject for this video:

Graphing

Reading:

- **General:** Section 4.2 Second Derivatives and Graphs
- **More Specifically:** pages 263 – 267, Examples 4,5,6

Homework:

H60: Graphing (4.2#45,49,57,77)

Useful Section 4.1 concepts discussed in previous videos

Correspondence between

sign behavior of $f'(x)$ at a particular $x = c$ and behavior of the graph of $f(x)$ at $x = c$

- If $f'(c)$ is *positive* then the line tangent to graph of $f(x)$ at $x = c$ *tilts upward*
- If $f'(c)$ is *negative* then the line tangent to graph of $f(x)$ at $x = c$ *tilts downward*
- If $f'(c)$ is *zero* then the line tangent to graph of $f(x)$ at $x = c$ is *horizontal*

Correspondence between

sign behavior of $f'(x)$ on an interval (a, b) and behavior of graph of $f(x)$ on the interval (a, b)

- If $f'(x)$ is *positive* on an interval (a, b) then $f(x)$ is *increasing* on the interval (a, b).
- If $f'(x)$ is *negative* on an interval (a, b) then $f(x)$ is *decreasing* on the interval (a, b).
- If $f'(x)$ is *zero* on an interval (a, b) then $f(x)$ is *constant* on the interval (a, b).

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Definition of *Local Maximum*

Words: *a **local maximum** for* $f(x)$.

Meaning: a y value $y = f(c)$ such that

- $f(x)$ is continuous on an interval (m, n) containing $x = c$
- The y value $f(c)$ is the *greatest* y value on the interval (a, b).

  That is, $f(c) \geq f(x)$ for all x in the interval (m, n).

Definition of *Local Minimum*

Words: *The* y *value* $f(c)$ *is a **local minimum** for* $f(x)$.

Meaning: a y value $y = f(c)$ such that

- $f(x)$ is continuous on an interval (m, n) containing $x = c$
- The y value $f(c)$ is the *least* y value on the interval (a, b).

  That is, $f(c) \leq f(x)$ for all x in the interval (m, n).

Definition of *Local Extremum*

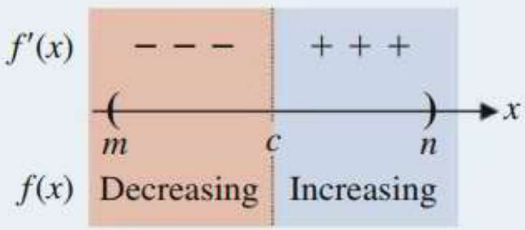
Words: *a **local extremum** for* $f(x)$.

Meaning: a y value $y = f(c)$ that is a ***local maximum*** or a ***local minimum***

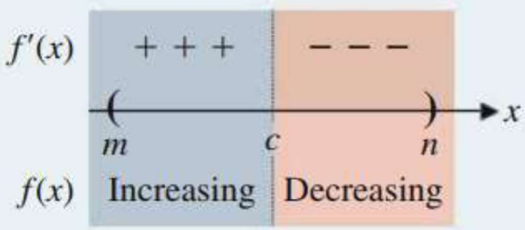
PROCEDURE First-Derivative Test for Local Extrema

Let c be a critical number of f [$f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ is not defined]. Construct a sign chart for $f'(x)$ close to and on either side of c.

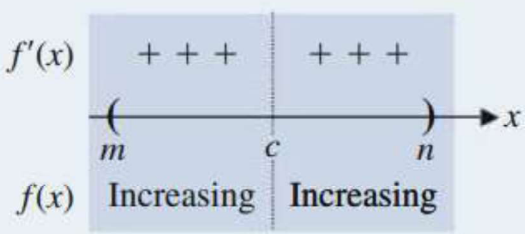
Sign Chart | **$f(c)$**



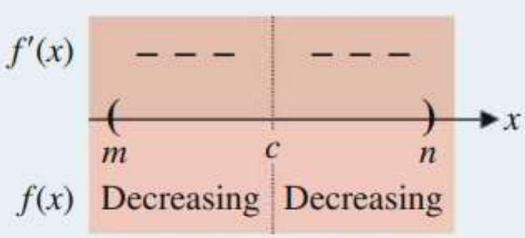
$f(c)$ is a local minimum.
If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a local minimum.



$f(c)$ is a local maximum.
If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a local maximum.



$f(c)$ is not a local extremum.
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.



$f(c)$ is not a local extremum.
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.

Useful Section 4.2 concepts discussed in previous videos

Correspondence between *sign behavior* of $f''(x)$ and *concavity behavior* of $f(x)$

- If $f''(x)$ is *positive* on an interval (a, b), then $f'(x)$ is *increasing* on the interval (a, b), which in turn means that $f(x)$ is *concave up* on the interval (a, b).
- If $f''(x)$ is *negative* on an interval (a, b), then $f'(x)$ is *decreasing* on the interval (a, b), which in turn means that $f(x)$ is *concave down* on the interval (a, b).

Definition of Concavity and Inflection Point

Words: f *is* ***concave up*** *on the interval* (a, b).

Graphical Definition: For every $x = c$, with $a < c < b$, the graph of f has a tangent line at $x = c$ and the graph of f stays above that tangent line for x-values in the interval (a, b).

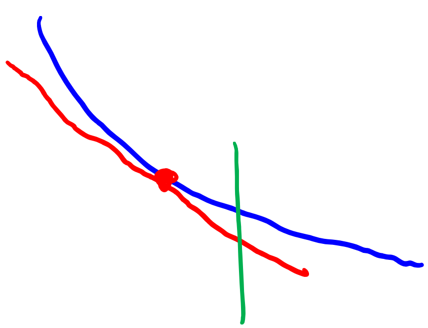
Abstract Definition: $f'(x)$ is *increasing* on the interval (a, b)

Words: f *is* ***concave down*** *on the interval* (a, b).

Graphical Definition: For every $x = c$, with $a < c < b$, the graph of f has a tangent line at $x = c$ and the graph of f stays below that tangent line for x-values in the interval (a, b).

Abstract Definition: $f'(x)$ is *increasing* on the interval (a, b)

Related terminology: An ***inflection point*** is point on the graph of a function where the function is continuous and the concavity changes (from up to down or from down to up.)



The correspondences

sign behavior of $f'(x)$ corresponds to increasing/decreasing behavior of $f(x)$

sign behavior of $f''(x)$ corresponds to concavity behavior of $f(x)$

can be used to articulate a strategy for graphing a function $f(x)$ given by a formula.

PROCEDURE Graphing Strategy (First Version)*

Step 1 *Analyze* $f(x)$. Find the domain and the intercepts. The x intercepts are the solutions of $f(x) = 0$, and the y intercept is $f(0)$.

Step 2 *Analyze* $f'(x)$. Find the partition numbers for f' and the critical numbers of f. Construct a sign chart for $f'(x)$, determine the intervals on which f is increasing and decreasing, and find the local maxima and minima of f.

Step 3 *Analyze* $f''(x)$. Find the partition numbers for $f''(x)$. Construct a sign chart for $f''(x)$, determine the intervals on which the graph of f is concave upward and concave downward, and find the inflection points of f.

Step 4 *Sketch the graph of f.* Locate intercepts, local maxima and minima, and inflection points. Sketch in what you know from steps 1–3. Plot additional points as needed and complete the sketch.

[Example 1] (Similar to 4.2#45) Given the following information

x	5	10	15
$f(x)$	12	8	4

$f'(5) = 0$ $\qquad$ $f'(15) = 0$

f': $- - - - -$ $\qquad$ f': $- - - - -$ $\qquad$ f': $+ + + + +$

sign chart for $f'(x)$ $\qquad$ $x = 5$ $\qquad$ $x = 15$ $\qquad$ x

f decreasing $\qquad$ f horiz tangent $\qquad$ f decreasing $\qquad$ f horiz tangent $\qquad$ f increasing

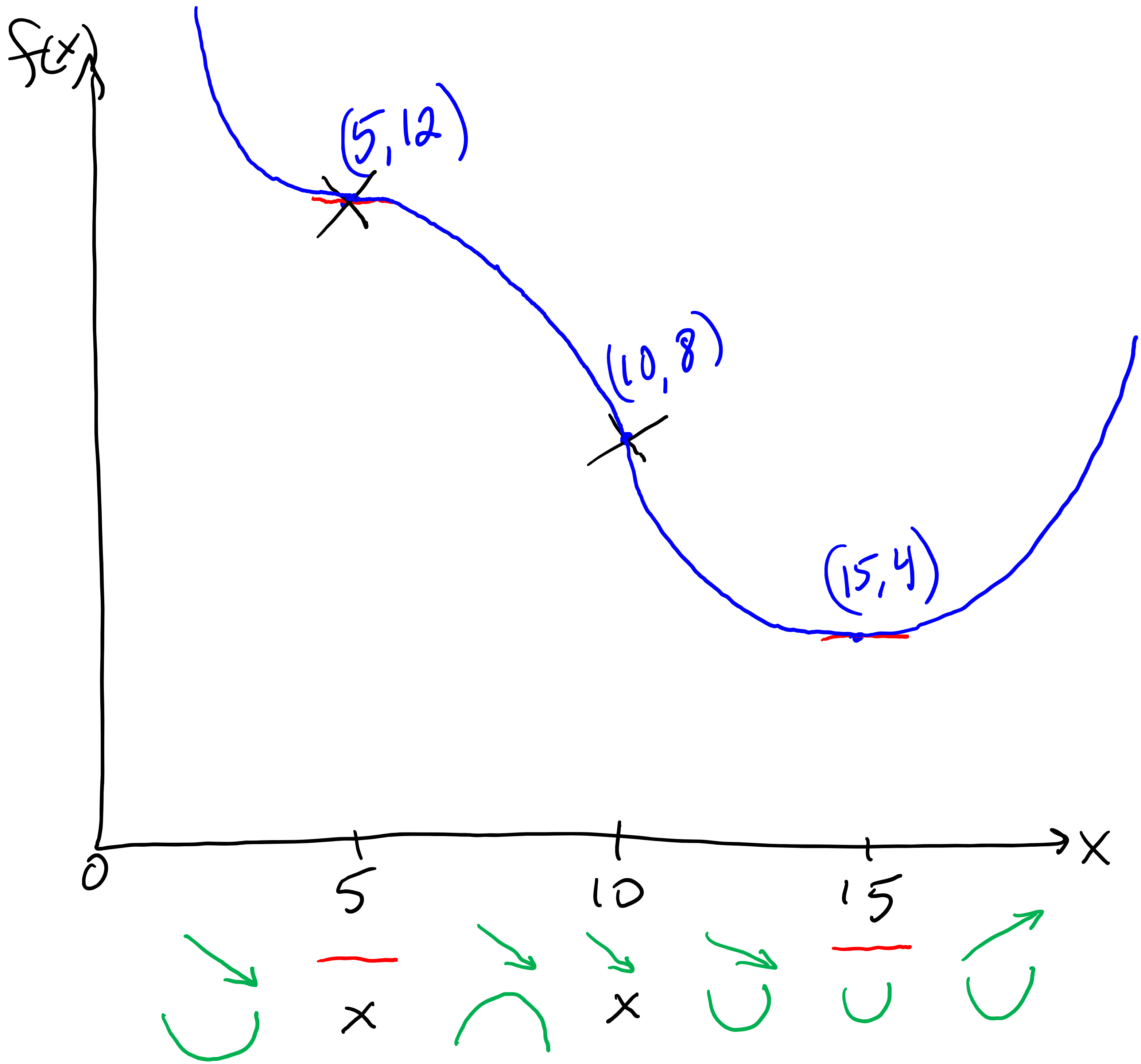
$f''(5) = 0$ $\qquad$ $f''(10) = 0$

f'': $+ + + +$ $\qquad$ f'': $- - - -$ $\qquad$ f'': $+ + + +$

sign chart for $f''(x)$ $\qquad$ $x = 5$ $\qquad$ $x = 10$ $\qquad$ x

f concave up $\qquad$ f has inflection $\qquad$ f concave down $\qquad$ f has inflection $\qquad$ f concave up

Sketch a possible graph of $f(x)$.


f(x)
(5, 12)
(10, 8)
(15, 4)
0
5
10
15
x

[Example 2] (Similar to 4.2#49) Given the following information

$$f(-2) = -2$$

$$f(0) = 1$$

$$f(2) = 4$$

$$f'(-2) = 0$$

$$f'(2) = 0$$

$$f'(x) > 0 \; on \; (-2,2)$$

$$f'(x) < 0 \; on \; (-\infty, -2) \; and \; (2, \infty)$$

$$f''(0) = 0$$

$$f''(x) > 0 \; on \; (-\infty, 0)$$

$$f''(x) < 0 \; on \; (0, \infty)$$

f horiz tangent at $x=2$

$f \nearrow$ on $(-2,2)$

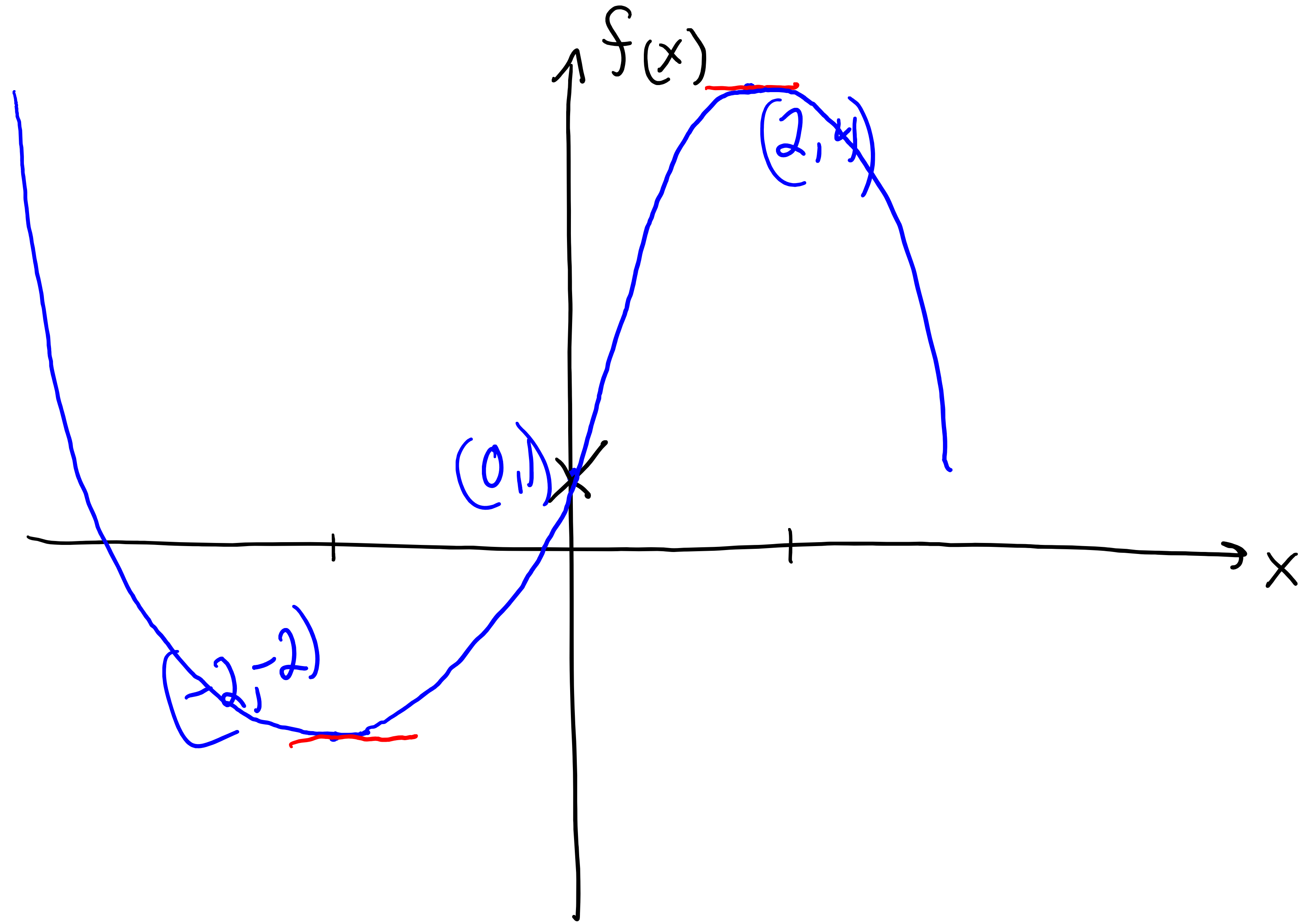
$f \searrow$ on $(-\infty,-2)$ and $(2,\infty)$

f inflection point at $x=0$

$f \cup$ on $(-\infty, 0)$

$f \cap$ on $(0, \infty)$

Sketch a possible graph of $f(x)$.



f(x)
(2, 4)
(0, 1)
(-2, -2)
x

[Example 3] (similar to 4.2#56) $f(x) = -x^4 + 6x^2 + 27$.

Use the graphing strategy to graph $f(x)$.

Solution

Step 1 Analyze $f(x)$

f is polynomial, so its domain is all real numbers

find y intercept by setting $x=0$ and finding y

$$y = f(0) = -(0)^4 + 6(0)^2 + 27 = 27$$

so y intercept at $(x,y) = (0,27)$

find x intercepts by setting $y=0$ and solving for x

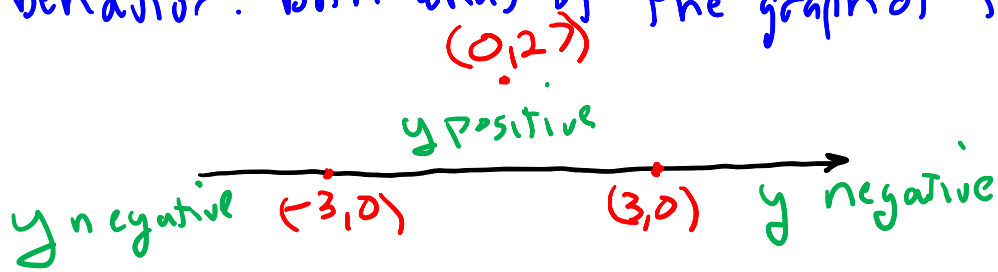
$$0 = y = f(x) = \underbrace{-x^4 + 6x^2 + 27}_{\text{standard form}} = \underbrace{-(x+3)(x-3)(x^2+3)}_{\text{factored form}}$$

factor

Solutions: $x=-3$, $x=3$

So x intercepts at $(x,y) = (-3,0)$ and $(x,y) = (3,0)$

End behavior: both ends of the graph of $f(x)$ go down.



Step 2 Analyze $f'(x) = \frac{d}{dx}(-x^4+6x^2+27) = -4x^3+12x =$

$a^2-b^2=(a+b)(a-b)$

$$= -4x(x^2-3) = -4x(x+\sqrt{3})(x-\sqrt{3})$$

$$= 4(x+\sqrt{3})\,x\,(x-\sqrt{3})$$

Partition numbers for $f'(x)$ are

$x=-\sqrt{3}$, $x=0$, $x=\sqrt{3}$

Sign chart for $f'(x)$

f' $+++$ | $f'(-\sqrt{3})=0$ | f' $- - -$ | $f'(0)=0$ | f' $+++$ | $f'(\sqrt{3})=0$ | f' $- - -$

$x=-\sqrt{3}$ $\quad$ $x=0$ $\quad$ $x=\sqrt{3}$

test $x=-2$ $\quad$ $x=-1$ $\quad$ $x=1$ $\quad$ $x=2$

$$f'(-2) = -4(-2)((-2)^2-3) = -4(-2)(4-3) = -4(-2)(1) = \text{pos}$$

$$f'(-1) = -4(-1)((-1)^2-3) = -4(-1)(1-3) = -4(-1)(-2) = \text{neg}$$

$$f'(1) = -4(1)((1)^2-3) = -4(1)(1-3) = -4(1)(-2) = \text{pos}$$

$$f'(2) = -4(2)((2)^2-3) = -4(2)(4-3) = -4(2)(1) = \text{neg}$$

f increasing on $(\infty,-\sqrt{3})$ and $(0,\sqrt{3})$ because f' pos

f decreasing on $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$ because f' neg

f has local max at $x=-\sqrt{3}$ and $x=\sqrt{3}$

because f' changes from $+$ to 0 to $-$

f has local min at $x=0$

because f' changes from $-$ to 0 to $+$

The y coordinates of the extrema are

$$f(-\sqrt{3}) = -(-\sqrt{3})^4+6(-\sqrt{3})^2+27 = -9+6(3)+27$$

$$= -9+18+27 = -9+45 = 36$$

$$f(0) = -(0)^4+6(0)^2+27 = 27$$

$$f(\sqrt{3}) = -(\sqrt{3})^2+6(\sqrt{3})^2+27 = -9+18+27 = 36$$

So local maxes at $(x,y) = (-\sqrt{3},36)$ and $(\sqrt{3},36)$, local min at $(0,27)$

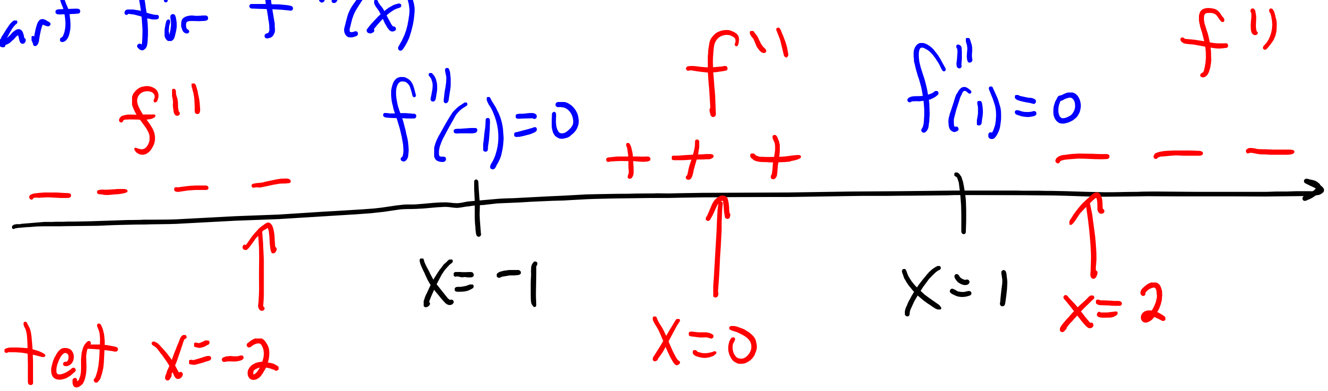
Step 3 Analyze $f''(x)$

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(-4x^3+12x\right) = -12x^2+12 =$$

$$= -12\left(x^2-1\right) = -12(x+1)(x-1)$$

Partition numbers for $f''(x)$ are $x=-1$, $x=1$

Sign chart for $f''(x)$



$$f''(-2) = -12\left((-2)^2-1\right) = -12(4-1) = -12(3) = \text{neg}$$

$$f''(0) = -12\left((0)^2-1\right) = -12(-1) = \text{pos}$$

$$f''(2) = -12\left((2)^2-1\right) = -12(4-1) = -12(3) = \text{neg}$$

f concave up on interval $(-1,1)$ because f'' pos

f concave down on intervals $(-\infty,-1)$ and $(1,\infty)$ because f'' is neg

f has inflection points at $x=-1$, $x=1$

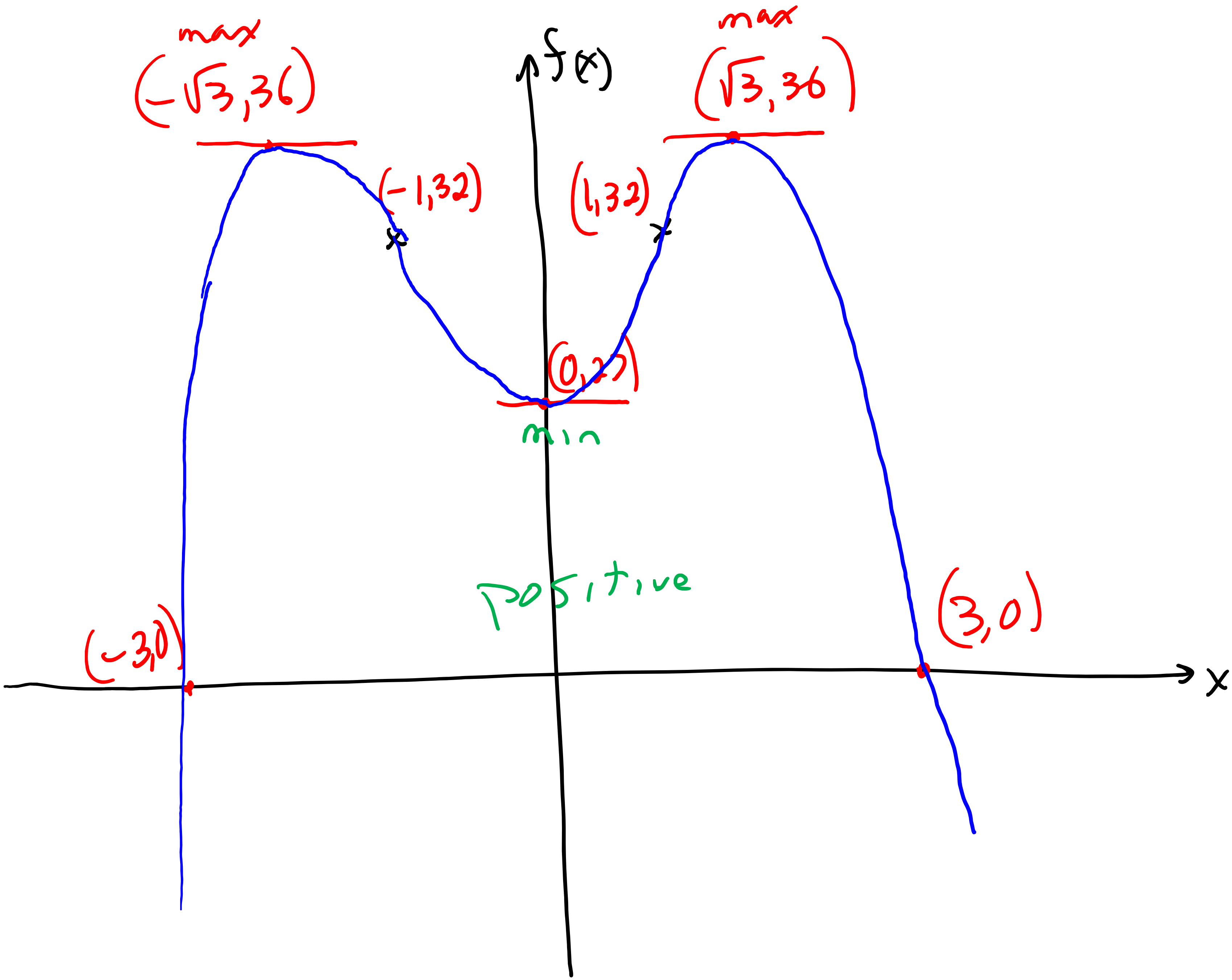
The y coordinates of the inflection points are

$f(-1) = -(-1)^4 + 6(-1)^2 + 27 = -(1) + 6(1) + 27 = 32$

$f(1) = -(1)^4 + 6(1)^2 + 27 = -(1) + 6(1) + 27 = 32$

So the (x,y) coordinates of the inflection points are

$(x,y) = (-1, 32)$ and $(x,y) = (1, 32)$

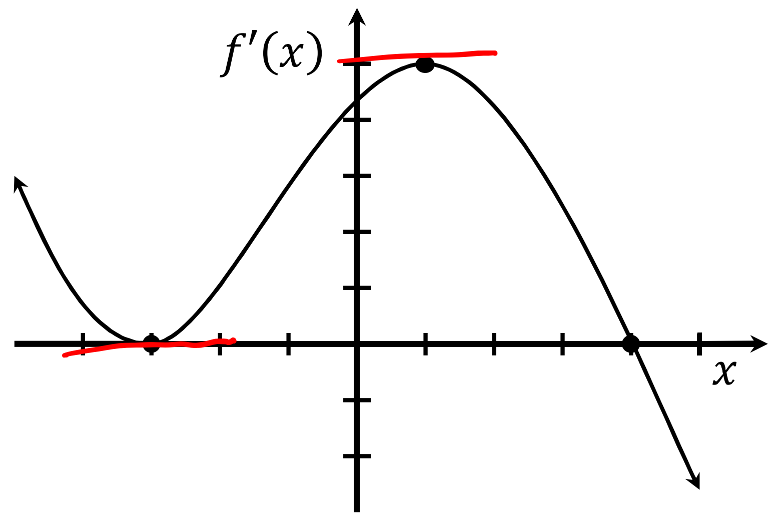


max
$(-\sqrt{3}, 36)$
max
$(\sqrt{3}, 36)$
$f(x)$
$(-1, 32)$
$(1, 32)$
$(0, 27)$
min
positive
$(-3, 0)$
$(3, 0)$
x

[Example 4] (similar to 4.2#77)

(revisiting an example from the videos for Homework H53 and H56)

The graph of f' is shown at right.



(A) Fill in the table below.

x	sign of $f'(x)$ (circle one)	incr/decr behavior of $f'(x)$ (circle one)	conclusions about behavior of function $f(x)$
$x < -3$	pos neg zero	incr decr horiz tan	f increasing f concave down
$x = -3$	pos neg zero	incr decr horiz tan	f horiz tan f inflection
$-3 < x < 1$	pos neg zero	incr decr horiz tan	f increasing f concave up
$x = 1$	pos neg zero	incr decr horiz tan	f increasing f inflection
$1 < x < 4$	pos neg zero	incr decr horiz tan	f increasing f concave down
$x = 4$	pos neg zero	incr decr horiz tan	f horiz tan f concave down
$4 < x$	pos neg zero	incr decr horiz tan	f decreasing f concave down

(B) Sketch a possible graph of $f(x)$ below.